ON THE MOTION OF A RIGID BODY WITH A LIQUID-FILLED CAVITY

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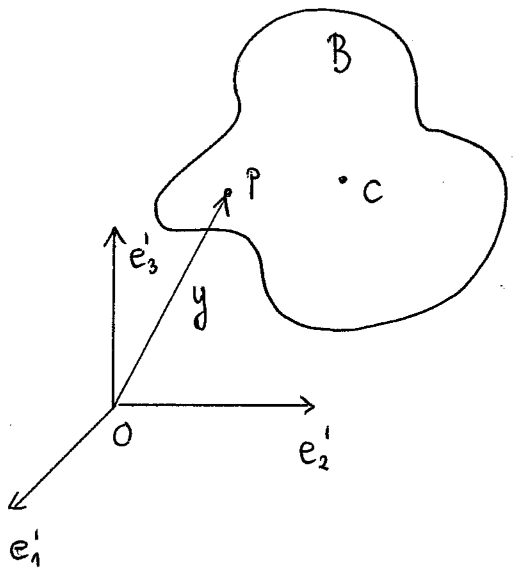
Lecture 1 (12.2.2018)



$\mathcal{E}$ is a system of external forces.

$\mathcal{C}$ is a center of mass.

$$\boxed{M\frac{dV}{dt} = F} \qquad \boxed{\frac{dK_0}{dt} = M_0} \qquad (1)$$

K_0 ... angular momentum

$$K_0 = \int_{B(t)} \tilde{\rho}_B(y)\, y \times v(y)\, dy\,, \quad v(y) = V_0 + \Omega \times y \quad (\text{Mozzi})$$

$$= \int_{B(t)} \tilde{\rho}_B(y)\, y \times (\Omega \times y)\, dy = \int_{B(t)} \tilde{\rho}(y)\left[|y|^2\Omega + \Omega\cdot y\, y\right]$$

$\mathbb{J}_0$ ... inertia tensor

$$\mathbb{J}_0 = \int_{B(t)} \rho_B(y) \left[|y|^2 \mathbb{1} - y \otimes y \right] dy$$

$$K_0 = \mathbb{J}_0 \cdot \Omega \quad \Longrightarrow \quad \frac{d}{dt} \left(\mathbb{J}_0 \cdot \Omega \right) = M_0 \qquad (2)$$

In order to write equation (2) in a body-fixed frame, we introduce

$$\mathbb{A} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

$\mathbb{A}$ ... rotation matrix

$$\dot{\mathbb{Q}} = \mathbb{A} \cdot \mathbb{Q} \;, \quad \mathbb{Q}(0) = \mathbb{1} \;, \quad t \geq 0$$

Since $\mathbb{A}$ is skew-symmetric:

(i) $\mathbb{Q}^T(t) \cdot \mathbb{Q}(t) = \mathbb{1} \;, \quad \forall t \geq 0$

(ii) $\det \mathbb{Q}(t) = 1 \;, \quad \forall t \geq 0$

Change of variables: $x = \mathbb{Q}(t) \cdot y$

Transformed angular velocity: $\omega = \mathbb{Q}^T \cdot \Omega$

$$\rho_B(x) = \tilde{\rho}_B(\mathbb{Q} \cdot x)$$

$$\mathbb{J}_0 = \int_{B(0)} \rho_B(x) \left[|x|^2 \mathbb{1} - \mathbb{Q} \cdot x \otimes x \cdot \mathbb{Q}^T \right] dx = \mathbb{Q} \cdot \mathbb{I}_0 \cdot \mathbb{Q}^T$$

$\mathbb{I}_0 = \mathbb{Q}^T \cdot \mathbb{J}_0 \cdot \mathbb{Q}$ ... defined in the body-fixed frame and therefore it is time-independent

Angular momentum equation in the body-fixed frame:

$$M_o = \frac{d}{dt}(\mathbb{J}_o \cdot \Omega) = \frac{d}{dt}\left(\mathbb{Q} \cdot \mathbb{I}_o \cdot \mathbb{Q}^T \cdot \Omega\right)$$

$$= \frac{d}{dt}\left(\mathbb{Q} \cdot \mathbb{I}_o \cdot \omega\right) = \left\{ \text{employing } \dot{\mathbb{Q}} = \mathbb{A} \cdot \mathbb{Q} \text{ and } \mathbb{Q}^T \cdot \mathbb{Q} = \mathbb{Q} \cdot \mathbb{Q}^T = \mathbb{1} \right\}$$

$$= \mathbb{Q} \cdot \left[\mathbb{I}_o \cdot \dot{\omega} + \omega \times (\mathbb{I}_o \cdot \omega) \right]$$

EULER EQUATION: $\boxed{\mathbb{I}_o \cdot \dot{\omega} + \omega \times (\mathbb{I}_o \cdot \omega) = \mathbb{Q}^T \cdot M_o}$

Inertia tensor in the body-fixed frame:

$$\mathbb{I}_o = \int_B \rho_B(x) \left[|x|^2 \mathbb{1} - x \otimes x \right] dx$$

Properties of $\mathbb{I}_o$:

(i) $\mathbb{I}_o$ is symmetric, i.e. $\mathbb{I}_o = \mathbb{I}_o^T$

(ii) $\mathbb{I}_o$ is positive-definite, i.e. $a \cdot \mathbb{I}_o \cdot a \geq 0 \,, \quad \forall a$

$$a \cdot \mathbb{I}_o \cdot a = 0 \text{ iff } a = 0$$

Lagrange identity: $|x|^2 a^2 - (a \cdot x)^2 = (a \times x)^2$

$\{e_1, e_2, e_3\}$ ... base constituted by eigenvectors of $\mathbb{I}_o$

$\{o, e_i\}$ ... principal frame (of inertia)

$\{c, e_i\}$ ... central frame

$(\mathbb{I}_0)_{ii}$ ... the moment of inertia of B with respect to $\{0, e_i\}$

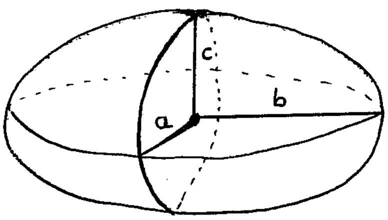
... eigenvalue corresponding to eigenvector e_i

$$(\mathbb{I}_0)_{11} = A = \int_B \rho_B(x)\,(x_2^2 + x_3^2)\,dx$$

$$(\mathbb{I}_0)_{22} = B = \int_B \rho_B(x)\,(x_1^2 + x_3^2)\,dx$$

$$(\mathbb{I}_0)_{33} = C = \int_B \rho_B(x)\,(x_1^2 + x_2^2)\,dx$$

Example. Ellipsoid



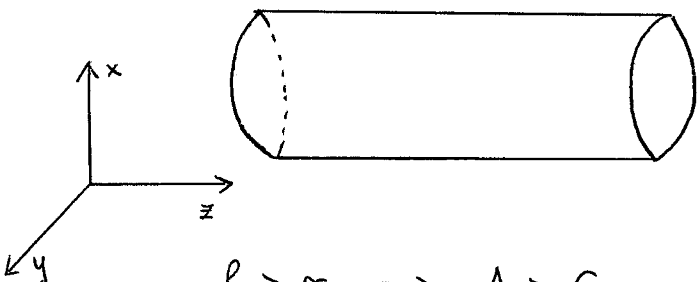
$$A = \frac{M}{5}(b^2+c^2)$$

$$B = \frac{M}{5}(a^2+c^2)$$

$$C = \frac{M}{5}(a^2+b^2)$$

$$\boxed{a < b < c \implies C < B < A}$$

Example. Cylinder (of radius r and length ℓ)



$$A = B = \frac{M}{12}(3r^2 + \ell^2)$$

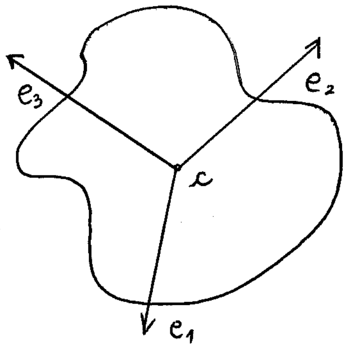
$$C = \frac{M}{2} r^2$$

$$\ell > r \implies A > C$$

Largest moment of inertia is about axis perpendicular to z-axis (cylinder axis).

$F = M_0 = 0$

$0 = c$



$\{c, e_i\}$ ... central frame

$\mathbb{I} = \mathbb{I}_c$

$\mathbb{I} \cdot \dot{\omega} + \omega \times (\mathbb{I} \cdot \omega) = 0$

$\dfrac{dK_c}{dt} = 0$

<u>Steady-state solution of Euler equation</u>

↳ permanent rotations $(\dot{\omega} = 0)$

$\omega \times (\mathbb{I} \cdot \omega) = 0$

$\boxed{\mathbb{I} \cdot \omega = \lambda \omega} \implies \lambda \in \{A, B, C\}$

$\omega = ke, \; e \in \varphi(\lambda)$ ... eigenspace associated to λ

Remark. $\dot{\omega} = 0 \implies \dot{\Omega} = 0$

$\omega_0^{(1)} = p_0 e_1 , \quad \omega_0^{(2)} = q_0 e_2 , \quad \omega_0^{(3)} = r_0 e_3$

Without loss of generality we can assume : $A \leq B \leq C$

(Lyapunov stabilities)

$\omega(t)$ is a generic solution to the Euler equation.

$\omega_0^{(i)}$ is stable if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|\omega(0) - \omega_0^{(i)}| < \delta \implies \sup_{t \geq 0} |\omega(t) - \omega_0^{(i)}| < \varepsilon .$$

$$\frac{dx}{dt} = f(x) \quad \text{in } \mathbb{R}^n, \quad f \text{ is Lipschitz}$$

$$f(0) = 0$$

$$V: B_R(0) \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}_+$$

(i) $V(0) = 0$

(ii) $V(x) > 0, \quad \forall x \in B_R$

(iii) $V(x(t)) \leq V(x(0)), \quad \forall t \geq 0$

and all solutions to (1) with initial data in $B_R(0)$.

$\Longrightarrow$ stability of $x = 0$

Kinetic energy: $T = \frac{1}{2} \omega \cdot \mathbb{I} \cdot \omega = \text{const}$.

Angular momentum: $|\mathbb{I} \cdot \omega| = \text{const}$.

Example. $A \leq B < C$

Is $\omega_0^{(3)} = r_0 e_3$ stable?

$$\omega = (p, q_0, r), \quad \omega^{(3)} = (p, q_0, r + r_0)$$

$$T = \frac{1}{2}\left(Ap^2 + Bq_0^2 + C(r+r_0)^2\right)$$

$$|\mathbb{I} \cdot \omega| = A^2p^2 + B^2q_0^2 + C^2(r+r_0)^2$$

$$V_1 = Ap^2 + Bq_0^2 + C(r^2 + 2rr_0) = \text{const}.$$

$$V_2 = A^2p^2 + B^2q_0^2 + C^2(r^2 + 2rr_0) = \text{const}.$$

$$CV_1 - V_2 = A(C-A)p^2 + B(C-B)q_0^2 = \text{const}.$$

$$CV_1 - V_2 + \frac{1}{4C^2r_0^2} V_2^2 = A(C-A)p^2 + B(C-B)q_0^2 + C^2r^2 + \text{H.O.T.}$$

Lyapunov function $\Longrightarrow$ rotation around C is stable

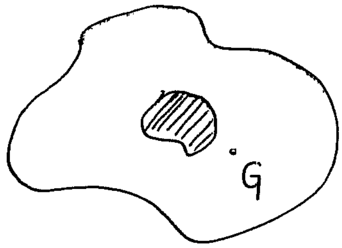
Suppose:

$A \le B < C$, $\omega_0 = r_0 e_3$ (stable)

$A < B \le C$, $\omega_0 = p_0 e_1$ (stable)

$A < B < C$, $\omega_0 = q_0 e_2$ (unstable)

Lecture 2 (13.2.2018)



Cavity inside a rigid body filled with liquid (Navier-Stokes)

- no forces acting on the coupled system

G = center of mass

NO-SLIP BOUNDARY CONDITIONS: continuity of velocity

Total angular momentum of the coupled system has to be constant in time!
(even though NS equations are dissipative)

$$\frac{d}{dt}\left(\mathbb{J} \cdot \overline{\omega} + \int_{\mathcal{C}(t)} \rho\, y \times w \right) = 0 \quad , \quad \mathcal{C}(t) \dots \text{cavity}$$

Equations of motion in a body-fixed frame (inviscid case)

Zhukovskiy

Cavity is simply connected.

$\operatorname{curl} u(x,t) = 0 \quad , \forall x, t$

$\boxed{u(x,t) = \nabla \varphi(x,t)}$ Zhukovskiy potential

(1) $\Delta\varphi(x,t) = 0 \quad \text{in } \mathcal{C}\times(0,\infty)$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\partial\mathcal{C}} = \omega\times x\cdot n \quad \text{on } \partial\mathcal{C}\times(0,\infty)$$

(2) $\varphi(x,t) = \omega_i(t)\,\psi_i(x)$

$$\begin{cases} \Delta\psi_i = 0 \quad \text{in } \mathcal{C} \\ \left.\dfrac{\partial\psi_i}{\partial n}\right|_{\partial\mathcal{C}} = e_i\times x\cdot n = x\times n\cdot e_i \end{cases}$$

$$K = \mathbb{J}\cdot\omega + \int_{\mathcal{C}} \rho x\times u$$

$$\int_{\mathcal{C}} \rho x\times\nabla\varphi = \omega_i\int_{\mathcal{C}} x\times\nabla\psi_i = \omega_i\int_{\mathcal{C}}\nabla\times(x\psi_i) = \omega_i\int_{\partial\mathcal{C}} x\times n\,\psi_i$$

$$= \omega_i\int_{\partial\mathcal{C}}\frac{\partial\psi_i}{\partial n}\,e_j\cdot\psi_i \qquad (\rho \text{ is missing everywhere})$$

$$(\mathbb{J}^*)_{ij} := \int_{\partial\mathcal{C}}\rho\,\psi_i\,\frac{\partial\psi_j}{\partial n} \qquad (\text{geometrical quantity; depends only on } \mathcal{C})$$

$$\Longrightarrow \int_{\mathcal{C}}\rho x\times\nabla\varphi = \mathbb{J}^*\cdot\omega \Longrightarrow K = (\mathbb{J}+\mathbb{J}^*)\cdot\omega$$

The motion of the liquid has been eliminated.

$$\begin{cases} \dot K + \omega\times K = 0 \\ K = (\mathbb{J}+\mathbb{J}^*)\cdot\omega \end{cases}$$

Properties of $\mathbb{J}^*$:

(i) $\mathbb{J}^*$ is symmetric

(ii) $\mathbb{J}^*$ is positive semi-definite

In the class of irrotational flow, the motion of the coupled system is reduced to the Euler equation for the equivalent body

$$\boxed{\mathbb{J}' \cdot \dot{\omega} + \omega \times (\mathbb{J}' \cdot \omega) = 0} \qquad \text{where } \mathbb{J}' = \mathbb{J} + \mathbb{J}^*.$$

STEADY-STATE SOLUTIONS

$v \in L^3_\sigma(\mathcal{E})$, $L^p_\sigma(\mathcal{E}) = \{ v \in L^p(\mathcal{E}) : \operatorname{div} v = 0,\ v \cdot n|_{\partial\mathcal{E}} = 0 \}$

$\Rightarrow v \in \mathcal{C}^\infty(\mathcal{E})$

$v \in W^{2,q}(\mathcal{E})$, $1 < q < \infty$

$\exists p \in C^\infty(\mathcal{E}) \cap W^{1,q}(\mathcal{E})$, $1 < q < \infty$

Steady-state solutions $\Rightarrow v \equiv 0$

$\Rightarrow \omega \times \mathbb{I} \cdot \omega = 0 \Rightarrow \boxed{\mathbb{I} \cdot \omega = \lambda \omega}$

$\mathbb{I}$ ... tensor of inertia of the coupled system

The class $\mathcal{S}$ of the steady-state solutions is of the form $(v = 0, \omega_0 = k e)$

$e \in \mathcal{S}(\lambda)$, $\lambda \in \{A, B, C\}$, A, B, C are eigenvalues of $\mathbb{I}$

$$K = \mathbb{I} \cdot \omega + \int_{\mathcal{E}} \rho x \times v$$

$$\omega^* := \mathbb{I}^{-1} \cdot K = \omega + \mathbb{I}^{-1} \cdot \int_{\mathcal{E}} \rho x \times v =: \omega - a$$

$$\boxed{\mathbb{I} \cdot \dot{\omega}^* + (\omega^* + a) \times (\mathbb{I} \cdot \omega^*) = 0}$$

Energy equality (with $\rho = 1$)

$$\frac{1}{2} \frac{d}{dt} \left(\|v\|_2^2 - a \cdot \mathbb{I} \cdot a + \omega^* \cdot \mathbb{I} \cdot \omega^* \right) = -\mu \|\nabla v(t)\|_2^2$$ (dissipation comes only from the liquid)

$$E(t) := \|v(t)\|_2^2 - a \cdot \mathbb{I} \cdot a$$

$\Rightarrow \exists c_0 \in (0,1)$ s.t. $\boxed{c_0 \|v\|_2^2 \leq E(t) \leq \|v\|_2^2}$

Lecture 3 (14.2.2018)

Steady-state $\mathcal{S}$

$S_0 \in \mathcal{S}$, $S_0 = (v=0, \omega=\omega_0)$

$\omega_0 \in \mathcal{S}(\lambda)$, $\lambda \in \{A, B, C\}$

$V=0$ ... liquid is in rest w.r.t. cavity

Lemma. There is $c_0 \in (0,1)$ such that

$$\mathbb{E}(v) := \|v\|_2^2 - a(v)\cdot \mathbb{I} \cdot a(v) \geq c_0 \|v\|_2^2$$

Proof. $K : L^2(\mathcal{C}) \longrightarrow \mathbb{R}^3$

$K(v) = a(v) \times x$, $a(v) = -\mathbb{I}^{-1} \cdot \int_{\mathcal{C}} x \times v$, $\mathbb{I} = \mathbb{I}_L + \mathbb{I}_B$

Compact, symmetric $(K(v), w) = (v, K(w))$, self-adjoint.

$M : L^2(\mathcal{C}) \to L^2(\mathcal{C})$

$$\boxed{M(v) = v + K(v)}$$

M is Fredholm operator of index 0

$\mathrm{Ker}(M) = 0 \rightsquigarrow Mv = 0 \Rightarrow v = 0$

$$(Mv, v) = \|v\|_2^2 - a(v)\cdot \mathbb{I} \cdot a(v)$$
$$= \|v - a \times x\|_2^2 + a(v) \cdot \mathbb{I}_B \cdot a(v)$$

M is invertible and self-adjoint.

$$0 \leq c_0 := \inf_{\|v\|_2 = 1} (M(v), v)$$

$c_0 \in \sigma(M) \Longrightarrow c_0 \neq 0$ since M is invertible $\Longrightarrow c_0 > 0$

$$\Longrightarrow (M(v), v) = \|v\|_2^2 - a(v) \cdot \mathbb{I} \cdot a(v) \geq c_0 \cdot \|v\|_2^2$$

$V = u - \omega \times x$

$\rho \|u\|_2^2 + \omega \cdot \mathbb{I} \cdot \omega$

$$\frac{d}{dt} E(t) = - \|D(u)\|_2^2$$

DEFINITION OF A WEAK SOLUTION

(v, ω_*) is a weak solution if:

(1) $v \in C_w(0,T; L^2_\sigma(\mathcal{C})) \cap L^2(0,T; W_0^{1,2}(\mathcal{C}))$

(2) (v, ω_*) satisfies strong energy inequality

$$E(t) + \omega_*(t) \cdot \mathbb{I} \cdot \omega_*(t) + \int_s^t \|\nabla v(t)\|_2^2 \leq E(s) + \omega_*(s) \cdot \mathbb{I} \cdot \omega_*(s) \ , \ \forall t \geq s,$$

$\forall$ a.a. $s \geq 0$ including $s = 0$

(3) $\omega_* \in C^1((0,T)) \cap C^0([0,T])$

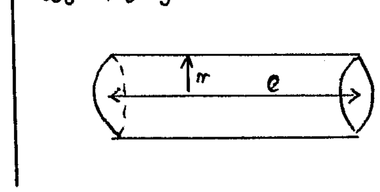
Existence: $\forall (v_0, (\omega_*)_0) \in L^2_\sigma(\mathcal{C}) \times \mathbb{R}^3$ there exists at least one weak solution such that

$$\lim_{t \to 0^+} \|v(t) - v_0\|_2 = \lim_{t \to 0^+} |\omega_*(t) - (\omega_*)_0| = 0 .$$

A, B, C ... principal moments of inertia of the whole system

In absence of liquid ω_0 is stable:

$\omega_0 = r_0 e_3$ (ω_0 is rotation around axis with maximal inertia)



$l \gg r$

$A \leq B < C$

$\omega_* = (p, q_0, r + r_0)$, $v + 0 = v$

$\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. $\|v(0)\|_2 + |p(0)| + |q(0)| + |r(0)| < \delta \implies$

$$\sup_{t \geq 0} \|v(t)\|_2 + |p(t)| + |q(t)| + |r(t)| < \varepsilon$$

No liquid : $V(t) = T + |\mathbb{I}\cdot\omega|^2$

Lemma. $U : B_R(0) \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ continuous

(i) $U(0) = 0$

(ii) $U(x) > 0, \quad x \in B_R(0)$

$F : L^2_\sigma(\mathcal{E}) \to \mathbb{R}_+$

$C_1 \|v\|_2 \leq F(v) \leq C_2 \|v\|_2$

$V := F(v) + U(\omega_*)$

If $V(t) \leq V(0)$, $\forall t \geq 0$, then s_0 is stable. ($s_0 = (0, r_0 e_3)$)

$$E(t) + Ap(t)^2 + Bq(t)^2 + C(r+r_0)(t)^2 \leq E(0) + Ap(0)^2 + Bq(0)^2 + C(r(0)+r_0)^2$$

$$\text{const} = |\mathbb{I}\cdot\omega_*|^2 = A^2p^2(t) + B^2q^2(t) + C^2(r(t)+r_0)^2$$

$$V = CV_1 - V_2 = CE(t) + C(C-A)\,p^2(t) + C(C-B)\,q^2(t) \leq V(0)$$

Spectral Stability Analysis

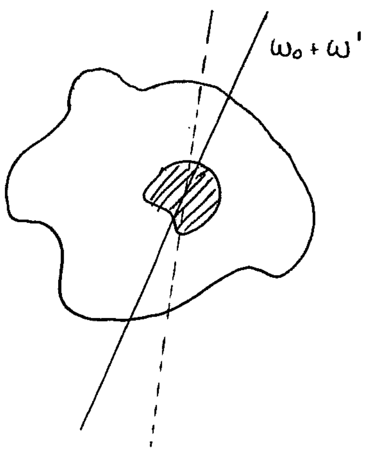
$u = (v,\omega)^T$

$\frac{du}{dt} + Lu = Nu$ in some Hilbert space

$\sigma(L) = \{\omega_0, A, B, C\}$

Stability : eigenvalues (real part) is positive, but 0 is also an eigenvalue

Luckily, $\sigma(L) \cap \mathrm{Im} = \{0\}$.



$P : L^2(\mathcal{E}) \to L^2_\sigma(\mathcal{E})$ ... Helmholtz projector

$$\frac{d}{dt}\underbrace{\begin{pmatrix} v + P(\omega\times x) \\ \mathbb{I}\cdot(\omega - a)\end{pmatrix}}_{I} + \underbrace{\begin{pmatrix} -\mu P\Delta v \\ \omega \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} P(\omega_0\times v) \\ \omega_0\times(\mathbb{I}\cdot\omega) + \omega\times(\mathbb{I}\cdot\omega_0) - \omega_0\times\mathbb{I}\cdot a - \omega \end{pmatrix}}_{B}$$

$$= -\underbrace{\begin{pmatrix} P\omega\times v + P(v\cdot\nabla v) \\ -\omega\times\mathbb{I}\cdot(\omega - a)\end{pmatrix}}_{N}$$

$$\boxed{H = L^2_\sigma(\mathcal{E})\times\mathbb{R}^3}$$

$u = (v,\omega)\in H$

$$Iu = \begin{pmatrix} v + P(\omega\times x) \\ \mathbb{I}\cdot(\omega - a)\end{pmatrix}$$

$$(A+B)(u) = 0 \implies \begin{cases} \left.\begin{array}{l} -\mu\Delta v = \omega_0\times v - \nabla p \\ \operatorname{div} v = 0 \end{array}\right\} \implies v = 0 \quad \text{because } v\big|_{\partial\mathcal{E}} = 0 \\ \omega_0\times\mathbb{I}\cdot\omega + \omega\times\mathbb{I}\cdot\omega_0 - \omega_0\times\mathbb{I}\cdot a = 0 \end{cases}$$

$v = 0 \implies a = 0$

We are left with: $\omega_0\times\mathbb{I}\cdot\omega + \omega\times\mathbb{I}\cdot\omega_0 = 0$

$$\omega_0\times\mathbb{I}\cdot\omega + \omega\times\lambda\omega_0 = 0$$

$$\omega_0\times(\mathbb{I}\cdot\omega - \lambda\omega) = 0$$

$$\mathbb{I}\cdot\omega - \lambda\omega = \sigma\omega_0 \ \big/ \cdot\omega_0$$

$$\omega_0\underbrace{(\mathbb{I}\cdot\omega - \lambda\omega)}_{=0} = \sigma|\omega_0|^2$$

$$\implies \sigma = 0 \implies \mathbb{I}\cdot\omega - \lambda\omega = 0$$

$$\underline{\underline{v = 0,\ \omega\in\mathcal{S}(\lambda)}}$$

Lecture 4 (15.2.2018)

$$\frac{d}{dt}u + I^{-1}(A+B)u = I^{-1}Nu$$

$\frac{d}{dt}u + Lu = Nu$ on X (Banach space)

$N(0) = 0$

Study the stability properties of $u \equiv 0$

$L = A + B$

A is the generator of an analytic semigroup.

A^{-1} exists and is compact

$\sigma(A)$ is discrete, $\mathrm{Re}(\sigma(A)) > 0$

Fractional power $A^{-\alpha}$, $A^{\alpha} = (A^{-\alpha})^{-1}$ are well defined for each $\alpha \in [0,\infty)$

$X_\alpha = \{u \in X : \|A^\alpha u\| \equiv \|u\|_\alpha < \infty\}$, $X_\alpha \subset X$

$\|Bu\| \leq c\|u\|_\alpha$, $\alpha \in [0,1)$

Properties of L:

1) $\sigma(L)$ is discrete.

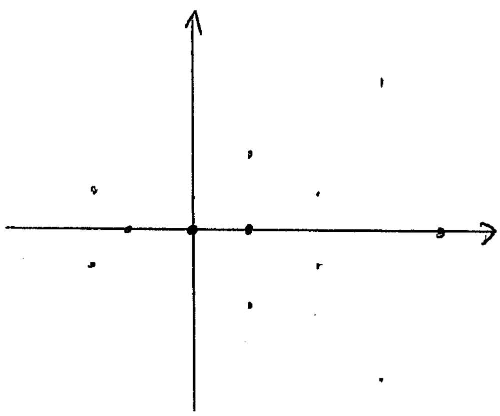
2) L is a generator of an analytic semigroup.

Assumptions:

1) $0 \in \sigma(L)$, $\dim \mathrm{Ker}(L) = m \geq 1$

2) $X = \mathrm{Ker}(L) \oplus \mathrm{Im}(L)$ ... the eigenvalue 0 is semi-simple.

$$\iff \mathrm{Ker}(L) \cap \mathrm{Im}(L) = \{0\}$$



$\sigma_0(L) = \{0\}$

$\sigma_1(L) = \sigma(L) \setminus \sigma_0(L)$

$P: X \to Ker(L)$

$Q = I - P : X \to Im(L)$

$L_0 := PL = LP \;,\; L_1 := QL = LQ \;,\; L = L_0 \oplus L_1$

$\sigma(L_0) = \sigma_0(L) \;,\; \sigma(L_1) = \sigma_1(L)$

$X = X_0 \oplus X_1$

) $\| N(u_1) - N(u_2) \| \leq C_\delta \| u_1 - u_2 \|_\alpha^p \;,\; p > 1 \;,\; \forall u_1, u_2 \in U_\delta(0) \subset X_\alpha$

$X = Ker(L) \oplus Im(L) \;,\; u = u^{(0)} + u^{(1)} \;,\; u^{(0)} \in Ker(L) \;,\; u^{(1)} \in Im(L)$

) $M(u^{(0)}, u^{(1)}) = N(u^{(0)} + u^{(1)})$

There is a continuous $\varepsilon(\rho)$, $\varepsilon(0) = 0$ such that

$$\| M(u^{(0)}, u^{(1)}) \| \leq \varepsilon(\rho) \| u^{(1)} \|_\alpha$$

whenever $\| u^{(0)} \| + \| u^{(1)} \| \leq c\rho$.

A general theorem (proof)

$$\frac{du}{dt} + Lu = Nu \quad / \; P \; (\text{projection})$$

$$\begin{cases} \dfrac{du^{(0)}}{dt} + L_0 u^{(0)} = PN(u) & \rightsquigarrow \; L_0 u^{(0)} = 0 \\ \dfrac{du^{(1)}}{dt} + L_1 u^{(1)} = QN(u) \end{cases}$$

$Re(\sigma(L_1)) \geq \gamma > 0$

$e^{-L_1 t}$ in $Im(L)$, $\| e^{-L_1 t} \| \leq c_0 e^{-\gamma t}$, $\| L_1^\alpha e^{-L_1 t} \| \leq c_1 \dfrac{e^{-\gamma t}}{t^\alpha}$

$L_1 = \hat{A} + \hat{B}$ in $Im(L)$

$L^\alpha \hat{A}^{-\alpha}$, $A^\alpha L^{-\alpha}$ are bounded in X

$\| L_1^\alpha u \|$, $\| \hat{A}^\alpha u \|$ equivalent

$$\begin{cases} u^{(1)}(t) = e^{-L_1 t} + \int_0^t e^{-L_1(t-s)} QN(u(s))\,ds \\ u^{(0)}(t) = u_0^{(0)} + \int_0^t PN(u^{(0)}(s) + u^{(1)}(s))\,ds \end{cases}$$

By the local theory, given $\rho > 0$ we can find $[0,t^*)$, $\eta > 0$ such that

$$\left\{\begin{array}{l} w := e^{bt} L_1^{\alpha} u^{(1)}, \quad 0 < b < \gamma \\ w(t) = e^{bt} e^{-L_1 t} L_1^{\alpha} u_0^{(1)} + e^{bt} \int_0^t L_1^{\alpha} e^{-L_1(t-s)} QM(u^{(0)}(s), e^{-bs} L_1^{-\alpha} w)\,ds \end{array}\right\}$$

$$\|u_0^{(0)}\| + \|w(0)\| < \eta \implies \|u^{(0)}(t)\| + \|w(t)\| < \rho, \ \forall t \in [0,\tau], \ \tau < t^*$$

$$t^* < \infty, \quad \lim_{t \to t^*} \|u(t)\|_{\alpha} = +\infty$$

$$\exists\, \tau_0 \in [0,t^*) : \|u^{(0)}(t)\| + \|w(t)\| < \rho \text{ and } \|u^{(0)}(\tau_0)\| + \|w(\tau_0)\| = \rho$$

$$\underline{\|w(t)\|} \leq c \|L_1^{\alpha} u_0^{(1)}\| + e^{bt} \int_0^t \|L_1^{\alpha} e^{-L_1(t-s)}\| \, \varepsilon(\rho) \, \|e^{-bs} L_1^{-\alpha} w\|_{\alpha}\, ds$$

$$\leq C_1 \|u_0\|_{\alpha} + \varepsilon(\rho) e^{bt} \int_0^t \frac{e^{-\gamma(t-s)}}{(t-s)^{\alpha}} e^{-bs} \|w(s)\|\,ds$$

$$\leq c_1 \eta + \varepsilon(\rho)\rho \int_0^{\infty} \frac{e^{-(\gamma-b)(t-s)}}{(t-s)^{\alpha}}\,ds$$

$$= \underline{c_1 \eta} + \underline{C_2 \varepsilon(\rho)\rho}$$

$$\|w(t)\| \leq c_1 \|u_0\|_{\alpha} + c_2 \varepsilon(\rho)\rho \leq c_1 \eta + c_2 \varepsilon(\rho)\rho, \quad t \in [0,\tau_0]$$

$$\|u^{(0)}(t)\| \leq \eta + \varepsilon(\rho) \int_0^t \|u^{(1)}(s)\|_{\alpha} ds \leq \eta + \varepsilon(\rho) c \int_0^t e^{-bs} \|w\| \leq \eta + c\varepsilon(\rho)\rho \int_0^{+\infty} e^{-bs} ds < \infty$$

$$\|u^{(1)}\|_{\alpha} = -e^{bt} \|L_1^{-\alpha} w\|_{\alpha} \leq C e^{-bt} \|w\|$$

$$\underline{\|u^{(0)}\| \leq \eta + C\varepsilon(\rho)\rho < \frac{\rho}{2}}$$

$$\|w(t)\| \leq Ce^{-(\gamma-b)t}\|u_0^{(1)}\|_\alpha + \varepsilon(\rho)\sup_{t\geq 0}\|w(t)\| \int_0 \cancel{\frac{e^{-(\gamma-b)(t-s)}}{(t-s)^\alpha}}\, ds$$

$$\sup_{t\geq 0}\|w(t)\| \leq C\|u_0^{(1)}\|_\alpha \Longrightarrow \|u^{(1)}(t)\|_\alpha \leq e^{-bt}\|u_0^{(1)}\|$$

$$\frac{du^{(0)}}{dt} = PM(u^{(0)}, u^{(1)})$$

$$\|u^{(0)}(t_1) - u^{(0)}(t_2)\| = \int_{t_1}^{t_2}\|PM(u^{(0)}, u^{(1)})\| \leq \varepsilon(\rho)\int_{t_1}^{t_2}\|u^{(1)}(t)\|_\alpha\, dt \leq C\int_{t_1}^{t_2} e^{-bt}dt$$

$\blacksquare$

Shatah & Strauss (Contemporary Math AMS Series, 2000.)

Lecture 5 (16.2.2018)

$$N[L] = \mathrm{Ker}(L) = \{(v,\omega) : v = 0,\ \omega = \omega^{(0)} \in \mathcal{G}(\lambda)\}$$

$$I^{-1}(A+B)u = u^{(0)},\quad u^{(0)} \equiv (0, \omega^{(0)})$$

(1) $A < B < C \quad \dim \mathcal{G}(\lambda) = 1$

(2) $A = B < C \quad \dim \mathcal{G}(\lambda) = 2$

(3) $A = B = C \quad \dim \mathcal{G}(\lambda) = 3$

) $\omega_0 \times [\mathbb{I}\cdot(\omega - a) - \lambda\omega] = \lambda\omega^{(0)}$

$$\mathbb{I} = \lambda \mathbb{1}$$

$$\omega_0 \times [\lambda(\omega - a) - \lambda\omega] = \lambda\omega^{(0)}$$

$$\omega_0 \times a = \omega^{(0)}$$

$$\omega^{(0)}\cdot a = 0 \rightsquigarrow \omega^{(0)}\cdot\int_{\mathcal{C}} x \times v = 0$$

$$-\mu\|\nabla v\|_2^2 = -\int_{\mathcal{C}} \omega^{(0)} \times x \cdot v = -\omega_0 \cdot \int_{\mathcal{C}} x \times v = 0 \Rightarrow v = 0$$

Nonlinear term:

$u = u^{(0)} + u^{(1)}$

$u^{(0)} = (0, \omega^{(0)})$, $\omega^{(0)} \in \mathcal{S}(\lambda)$

$u^{(1)} = (v, \omega^{(1)})$

First part: $-P[2(\omega^{(0)} + \omega^{(1)}) \times v + v \cdot \nabla v]$

Second part: $(\omega^{(0)} + \omega^{(1)}) \times \mathbb{I} \cdot (\omega^{(0)} + \omega^{(1)} - a)$

Crucial: $\omega^{(0)} \times \mathbb{I} \cdot \omega^{(0)} = 0$

$|\omega^{(0)} + \omega^{(1)}| \, \|v\|_2 + \|v \cdot \nabla v\|_2 + |\omega^{(1)}| \left(|\omega^{(0)}| + |\omega^{(1)}| + \|v\|_2 \right)$

$\|v \cdot \nabla v\|_2 \leq \|v\|_\infty \|\nabla v\|_2 \leq C \|v\|_\infty \|A_0^{1/2} v\|_2$

Kato, Fujita (1970): $\|v\|_\infty \leq c \|A_0^\alpha v\|_2$, $\alpha \in [\frac{3}{4}, 1)$

$\Longrightarrow \|v \cdot \nabla v\|_2 \leq C \|A_0^\alpha v\|_2 \|A_0^{1/2} v\|_2 \leq C \|A_0^\alpha v\|_2^2$.

<u>Linear energy analysis</u>

$$\frac{du}{dt} + \mathcal{L}u = 0 \qquad (\mathcal{L}E)$$

$s_0 = (0, \omega_0 = \omega_0 e)$, $e \in \mathcal{S}(\lambda)$

$h \in \mathbb{R}^3 \rightsquigarrow h = h_\perp + h_\parallel$, $h_\parallel = h_\parallel e$, $h_\perp \cdot e = 0$

$\boxed{\sigma(L) \cap \{i\mathbb{R}\} = \{0\}}$

$\exists \; i\zeta \in \sigma(L)$, $\zeta \neq 0$

$(\mathcal{L}E)$ must have a T-periodic solution such that $\int_0^T u(s)ds = 0$

$\omega_\parallel = a_\parallel + \text{const.} \Longrightarrow \omega_\parallel = 0$

$\int_0^T \|\nabla v(s)\|_2^2 = 0$

$v = 0$

$\text{curl}(\dot{\omega} \times x) = \text{curl}(\nabla p) = 0 = 2\dot{\omega}$

$\Longrightarrow \omega_\perp = 0$