Chapter 2

The Equations of Motion of a Rigid Body with a Liquid-Filled Cavity

Objective of this chapter is to derive the basic equations governing the motion of the *coupled* system constituted by a rigid body containing an interior cavity *entirely* filled with an incompressible fluid (in short: a liquid), and moving under the action of a given system of external forces.

2.1 Equations of Motion in an Inertial Frame

Let $\mathscr{B}$ be a rigid body moving with respect to the inertial frame $\mathcal{I} = \{O', \boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3'\}$, under the action of a system, Σ, of *external* active forces. We denote by $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{M}}_O$, in the order, resultant force and torque with respect to O of Σ. With a view to the applications we have in mind, *we suppose that both* $\boldsymbol{\mathcal{F}}$ *and* $\boldsymbol{\mathcal{M}}_O$ *are, at most, functions of time*. Likewise, in cases when $\mathscr{B}$ is constrained, we shall indicate by $\boldsymbol{\Phi}'$ and $\boldsymbol{\tau}_O'$ the (unknown) resultant force and torque with respect to O of the system of reaction forces. In order to simplify the presentation, the point O is chosen rigidly fixed to $\mathscr{B}$.

We now assume that $\mathscr{B}$ contains an *interior cavity entirely filled with a liquid*, $\mathscr{L}$, of constant density ρ. Again with a view to the applications, we assume that the body force acting on $\mathscr{L}$ is potential-like, and denote it by ∇U, for some (smooth) scalar function $U = U(x)$.

The main goal of this work is to investigate the motion of the *coupled system* $\mathscr{S} := \mathscr{B} \cup \mathscr{L}$.

Let $\boldsymbol{\eta}_O = \boldsymbol{\eta}_O(t)$ and $\boldsymbol{\varpi} = \boldsymbol{\varpi}(t)$ be, in the order, velocity of the point O and angular velocity of $\mathscr{B}$ in $\mathcal{I}$. Also, denote by $\mathscr{C} = \mathscr{C}(t)$ the region occupied by the cavity at time $t \geq 0$, $S(t) := \partial\mathscr{C}(t)$ its boundary, and by $\boldsymbol{w} = \boldsymbol{w}(y,t)$, $p = p(y,t)$, $y \in \mathscr{C}(t)$, Eulerian (absolute) velocity and pressure fields of $\mathscr{L}$. We thus have that balance of linear momentum and conservation of mass for $\mathscr{L}$ require that the generic motion of $\mathscr{L}$ in $\mathcal{I}$ be governed by the following set of equations

$$\left.\begin{aligned} &\rho\Big(\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{w}\cdot\nabla\boldsymbol{w}\Big) = \operatorname{div}\mathbb{T} \\ &\operatorname{div}\boldsymbol{w} = 0 \end{aligned}\right\} \quad (y,t) \in \bigcup_{t>0} \mathscr{C}(t)\times\{t\} \tag{2.1}$$

with $\mathbb{T} = \mathbb{T}(\boldsymbol{w}, p)$ (modified) Cauchy stress tensor given by

$$\mathbb{T} = \mu\,\mathbb{D}(\boldsymbol{w}) - (p-U)\,\widehat{\mathbb{I}}\,, \quad \mathbb{D}(\boldsymbol{w}) := \tfrac{1}{2}\left(\nabla\boldsymbol{w} + \nabla\boldsymbol{w}^\top\right), \tag{2.2}$$

and where $\mu \geq 0$ is the (constant) shear viscosity coefficient of $\mathscr{L}$. To the system (2.1), (2.2) we need to append appropriate boundary conditions at S. If $\mathscr{L}$ is viscous ($\mu > 0$) we will assume "no-slip" conditions, which in our case take the form

$$\boldsymbol{w}(y,t) = \boldsymbol{\eta}_O + \boldsymbol{\varpi}\times(y-y_O)\,, \quad (y,t)\in\bigcup_{t>0} S(t)\times\{t\}, \quad \text{if } \mu > 0\,, \tag{2.3}$$

with $y_O = y_O(t)$ position occupied by O, whereas, if $\mathscr{L}$ is inviscid ($\mu = 0$) we shall require

$$\boldsymbol{w}(y,t)\cdot\boldsymbol{N} = (\boldsymbol{\eta}_O + \boldsymbol{\varpi}\times(y-y_O))\cdot\boldsymbol{N}\,, \quad (y,t)\in\bigcup_{t>0} S(t)\times\{t\}, \quad \text{if } \mu = 0\,, \tag{2.4}$$

with $\boldsymbol{N} = \boldsymbol{N}(t)$ outer unit normal to $S(t)$.

As for the body $\mathscr{B}$, if we also take into account possible constraints, its motion is ruled by the following general equations of balance of linear momentum:

$$M_{\mathscr{B}}\,\dot{\boldsymbol{\eta}}_C = \boldsymbol{\mathcal{F}} + \boldsymbol{\Phi}' - \int_{S(t)} \mathbb{T}\cdot\boldsymbol{N}\,, \tag{2.5}$$

and angular momentum:

$$\frac{d}{dt}\big[\widehat{\mathbb{J}}_O\cdot\boldsymbol{\varpi} + M_{\mathscr{B}}(y_C - y_O)\times\boldsymbol{\eta}_O\big] + M_{\mathscr{B}}\,\boldsymbol{\eta}_O\times\boldsymbol{\eta}_C = \boldsymbol{\mathcal{M}}_O + \boldsymbol{\tau}'_O - \int_{S(t)} (y-y_O)\times\mathbb{T}\cdot\boldsymbol{N}\,. \tag{2.6}$$

In these equations, $M_{\mathscr{B}}$ is the mass of $\mathscr{B}$, $\boldsymbol{\eta}_C$ the velocity of its center of mass C, and $\widehat{\mathbb{J}}_O = \widehat{\mathbb{J}}_O(t)$ its inertia tensor with respect to O, whose components in $\mathcal{I}$ are given by

$$(\widehat{\mathbb{J}}_O)'_{ij} = \int_{\mathscr{B}(t)} \rho_{\mathscr{B}}[\boldsymbol{e}'_i\times(y-y_O)]\cdot[\boldsymbol{e}'_j\times(y-y_O)]\,, \quad i,j = 1,2,3\,, \tag{2.7}$$

with $\rho_{\mathscr{B}}$ material density of $\mathscr{B}$ and $\mathscr{B}(t)$ spatial region occupied by $\mathscr{B}$ at time t. Moreover, the last term on the right-hand side of (2.5) and (2.6) represents the (internal to $\mathscr{S}$) force and torque, respectively, exerted by the liquid on $\mathscr{B}$.

We shall next rewrite the above governing equations in an equivalent form that better describes the physical aspect of the problem. To this end, we begin to formally integrate $(2.1)_1$ over $\mathscr{C}(t)$, apply Reynolds transport theorem (e.g. [4, p. 78]), integrate by parts and use $(2.1)_2$ to obtain

$$\frac{d}{dt}\int_{\mathscr{C}(t)} \rho\, \boldsymbol{w} = \int_{S(t)} \mathbb{T}\cdot \boldsymbol{N}\,,$$

which, in turn, once replaced in (2.5), delivers

$$M\, \dot{\boldsymbol{\eta}}_G = \boldsymbol{\mathcal{F}} + \boldsymbol{\Phi}'\,, \tag{2.8}$$

with $M := M_{\mathcal{B}} + \int_{\mathscr{C}(t)} \rho$ total mass of the system liquid-body, $\mathscr{S}$, and $\boldsymbol{\eta}_G$ velocity of its center of mass G. In deriving (2.8) we have used the well-known relation

$$M_{\mathscr{B}}\, \boldsymbol{\eta}_C + \int_{\mathscr{C}(t)} \rho\, \boldsymbol{w} = M\, \boldsymbol{\eta}_G\,. \tag{2.9}$$

Equation (2.8) describes the motion of G in $\mathcal{I}$. Similarly, by cross-multiplying both sides of $(2.1)_1$ by $y - y_O$, we get

$$\int_{\mathscr{C}(t)} (y-y_O)\times \big[\rho\, \frac{d\boldsymbol{w}}{dt} - \operatorname{div}\mathbb{T}\big] = \boldsymbol{0}\,, \tag{2.10}$$

where $d/dt := \partial/\partial t + \boldsymbol{w}\cdot\nabla$ denotes material derivative. Since $\boldsymbol{w} = d\boldsymbol{y}/dt$ and $\boldsymbol{\eta}_O = d\boldsymbol{y}_O/dt$, we show

$$\int_{\mathscr{C}(t)} (y-y_O)\times \rho\, \frac{d\boldsymbol{w}}{dt} = \int_{\mathscr{C}(t)} \rho \frac{d}{dt}\big[(y-y_O)\times \boldsymbol{w}\big] + \boldsymbol{\eta}_O \times \int_{\mathscr{C}(t)} \rho\, \boldsymbol{w}\,,$$

whereas, by Gauss theorem and the symmetry property of $\mathbb{T}$,

$$\int_{\mathscr{C}(t)} (y-y_O)\times \operatorname{div}\mathbb{T} = \int_{S(t)} (y-y_O)\times \mathbb{T}\cdot\boldsymbol{N}\,.$$

From the last two displayed equations, with the help again of Reynolds theorem we infer

$$\frac{d}{dt}\int_{\mathscr{C}(t)} \rho\,(y-y_O)\times \boldsymbol{w} + \boldsymbol{\eta}_O \times \int_{\mathscr{C}(t)} \rho\, \boldsymbol{w} = \int_{S(t)} (y-y_O)\times \mathbb{T}\cdot\boldsymbol{N}\,. \tag{2.11}$$

Therefore, by (2.11) and (2.6) we conclude

$$\begin{aligned}\frac{d}{dt}\big[\widehat{\mathbb{J}}_O\cdot\boldsymbol{\varpi} + M_{\mathscr{B}}(y_C - y_O)\times \boldsymbol{\eta}_O + \int_{\mathscr{C}(t)} \rho\,(y-y_O)\times \boldsymbol{w}\big] + M\, \boldsymbol{\eta}_O\times\boldsymbol{\eta}_G \\ = \boldsymbol{\mathcal{M}}_O + \boldsymbol{\tau}'_O\,,\end{aligned} \tag{2.12}$$

which represents the equation of balance of the *total* angular momentum of the system $\mathscr{S}$.

Clearly (2.1)–(2.3), (2.8) and (2.12) are formally *equivalent* to the equations (2.1)–(2.3), (2.5), (2.6). Our main objective will be the investigation of the relevant properties of their solutions.

2.2 Equations of Motion in a Body-Fixed Frame

We observe a somehow undesired feature of (2.1)–(2.3), (2.8) and (2.12), namely, they are written in the inertial frame $\mathcal{I}$ where the location of the cavity $\mathscr{C}$ is time-dependent. Thus, following the classical procedure employed in the study of rigid-body dynamics and also adopted in certain problems of liquid-solid interaction similar to the one treated here (e.g. [3]), we shall rewrite the relevant equations with respect to a body-fixed frame. To this end, denote by $\mathcal{R} = \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ a frame with the origin in O and axes $\{\boldsymbol{e}_i\}$ attached to $\mathscr{B}$. Let $\{\varpi_i'\}$ be the components of $\boldsymbol{\varpi}$ in $\mathcal{I}$, set

$$\mathbb{A} := \begin{bmatrix} 0 & -\varpi_3' & \varpi_2' \\ \varpi_3' & 0 & -\varpi_1' \\ -\varpi_2' & \varpi_1' & 0 \end{bmatrix}$$

and consider the one-parameter family of tensors $\mathbb{Q} = \mathbb{Q}(t)$ solutions to the following system

$$\dot{\mathbb{Q}} = \mathbb{A} \cdot \mathbb{Q}\,, \quad \mathbb{Q}(0) = \widehat{\mathbb{I}}\,, \quad t \geq 0\,. \tag{2.13}$$

Since $\mathbb{A}$ (and, consequently, $\mathbb{Q}^\top \cdot \mathbb{A} \cdot \mathbb{Q}$) is skew-symmetric, from (2.13) it easily follows that (e.g. [5, Lemma 1.5 in Chapter III])

$$\mathbb{Q}(t) \cdot \mathbb{Q}^\top(t) = \mathbb{Q}^\top(t) \cdot \mathbb{Q}(t) = \widehat{\mathbb{I}}\,, \quad \det \mathbb{Q}(t) = 1\,, \quad \text{all } t \geq 0\,,$$

which means that $\mathbb{Q}(t)$ is proper orthogonal at each $t \geq 0$. We then introduce the change of variables

$$\boldsymbol{x} = \mathbb{Q}^\top(t) \cdot (y - y_O)$$

bringing the point $y \in \mathcal{I}$ into $x \in \mathcal{R}$. Observe that, in this transformation, namely, when referred to the frame $\mathcal{R}$, the cavity becomes the time-independent region, $\mathcal{C}$, given by

$$\mathcal{C} := \left\{x \in \mathbb{R}^3 : \ \boldsymbol{x} = \mathbb{Q}^\top(t) \cdot (y - y_O(t))\,,\ y \in \mathscr{C}(t)\,,\ t \geq 0\right\}\,.$$

If, without loss of generality, we take $\mathcal{I} \equiv \mathcal{R}$ at time $t = 0$, it then follows that $\mathcal{C} \equiv \mathscr{C}(0)$. Likewise, in the frame $\mathcal{R}$ the region occupied by the body $\mathscr{B}$ becomes,

$$\mathcal{B} := \left\{x \in \mathbb{R}^3 : \ \boldsymbol{x} = \mathbb{Q}^\top(t) \cdot (y - y_O(t))\,,\ y \in \mathscr{B}(t)\,,\ t \geq 0\right\}\,,$$

with $\mathcal{B} \equiv \mathscr{B}(0)$.

Assumption. *Throughout this article, we assume* $\mathcal{B} := \Omega_1 \setminus \overline{\Omega}_2$, $\mathcal{C} := \Omega_2$, *where* Ω_i, $i = 1, 2$, *are bounded domains in* $\mathbb{R}^3$ *with* $\overline{\Omega}_2 \subset \Omega_1$.

Define (with P denoting any point rigidly fixed to $\mathscr{B}$ and $\boldsymbol{\eta}_P$ its velocity)

$$\begin{aligned}&\boldsymbol{u}(x,t) := \mathbb{Q}^\top(t)\cdot\boldsymbol{w}(\mathbb{Q}(t)\cdot\boldsymbol{x}+\boldsymbol{y}_O(t),t)\,,\quad \widetilde{p}(x,t) := (p-U)(\mathbb{Q}(t)\cdot\boldsymbol{x}+\boldsymbol{y}_O(t),t)\\ &\mathbb{J}_O := \mathbb{Q}^\top(t)\cdot\widehat{\mathbb{J}}_O\cdot\mathbb{Q}(t)\,,\quad \boldsymbol{\xi}_P(t) := \mathbb{Q}^\top(t)\cdot\boldsymbol{\eta}_P(t)\,,\quad \boldsymbol{\omega}(t) := \mathbb{Q}^\top(t)\cdot\boldsymbol{\varpi}(t)\\ &\boldsymbol{v}(x,t) := \boldsymbol{u}(x,t)-\boldsymbol{\xi}_O(t)-\boldsymbol{\omega}(t)\times\boldsymbol{x}\quad \text{(relative velocity of } \mathscr{L} \text{ in } \mathcal{R})\\ &\boldsymbol{F}(t) := \mathbb{Q}^\top(t)\cdot\boldsymbol{\mathcal{F}}(t)\,,\quad \boldsymbol{M}_O(t) := \mathbb{Q}^\top(t)\cdot\boldsymbol{\mathcal{M}}_O(t)\,,\\ &\boldsymbol{\Phi} := \mathbb{Q}^\top(t)\cdot\boldsymbol{\Phi}'\,,\quad \boldsymbol{\tau}_O := \mathbb{Q}^\top(t)\cdot\boldsymbol{\tau}'_O\,,\quad \boldsymbol{n} := \mathbb{Q}^\top(t)\cdot\boldsymbol{N}(t)\,.\end{aligned} \tag{2.14}$$

Furthermore, setting $\boldsymbol{a} := \mathbb{Q}(t)\cdot\boldsymbol{b}$, for arbitrary $\boldsymbol{b}\in\mathbb{R}^3$, from (2.13), (2.14) and the identity $\mathbb{Q}^\top(t)\cdot(\boldsymbol{\varpi}\times\boldsymbol{a}) = \boldsymbol{\omega}\times\boldsymbol{b}$, we deduce $\mathbb{Q}^\top(t)\cdot\dot{\mathbb{Q}}(t)\cdot\boldsymbol{b} = \mathbb{B}\cdot\boldsymbol{b}$ with

$$\mathbb{B} := \begin{bmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}, \tag{2.15}$$

and $\{\omega_i\}$ components of $\boldsymbol{\omega}$ in $\mathcal{R}$. By the arbitrariness of $\boldsymbol{b}$ we then infer that $\mathbb{Q}$ must satisfy

$$\dot{\mathbb{Q}} = \mathbb{Q}\cdot\mathbb{B}\,,\quad \mathbb{Q}(0) = \widehat{\mathbb{I}}\,. \tag{2.16}$$

Using this property and (2.14), by a straightforward calculation (see [3, p. 667-669] for details) one can then show that equations (2.1)–(2.3), (2.8), (2.12) when written in the new variables (that is, in the frame $\mathcal{R}$) assume the following form

$$\begin{aligned}&\left.\begin{aligned}&\rho\Big(\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{u}+\boldsymbol{\omega}\times\boldsymbol{u}\Big) = \mu\,\Delta\boldsymbol{u}-\nabla\widetilde{p}\\ &\operatorname{div}\boldsymbol{u} = 0\end{aligned}\right\}\quad (x,t)\in\mathcal{C}\times(0,\infty)\\ &\left.\begin{aligned}&\boldsymbol{u}(x,t) = \boldsymbol{\xi}_O(t)+\boldsymbol{\omega}(t)\times\boldsymbol{x}\,,\quad (\text{if } \mu>0)\\ &\boldsymbol{u}(x,t)\cdot\boldsymbol{n} = (\boldsymbol{\xi}_O(t)+\boldsymbol{\omega}(t)\times\boldsymbol{x})\cdot\boldsymbol{n}\,,\quad (\text{if } \mu=0)\end{aligned}\right\}\quad (x,t)\in\partial\mathcal{C}\times(0,\infty)\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&M\,\dot{\boldsymbol{\xi}}_G+M\,\boldsymbol{\omega}\times\boldsymbol{\xi}_G = \boldsymbol{F}+\boldsymbol{\Phi}\\ &\dot{\boldsymbol{A}}_O+\boldsymbol{\omega}\times\boldsymbol{A}_O+M\,\boldsymbol{\xi}_O\times\boldsymbol{\xi}_G = \boldsymbol{M}_O+\boldsymbol{\tau}_O\,,\end{aligned} \tag{2.18}$$

with

$$\boldsymbol{A}_O := \mathbb{J}_O\cdot\boldsymbol{\omega}+M_{\mathcal{B}}\boldsymbol{x}_C\times\boldsymbol{\xi}_O+\int_{\mathcal{C}}\rho\,\boldsymbol{x}\times\boldsymbol{u}\,, \tag{2.19}$$

and $M_{\mathcal{B}}\equiv M_{\mathscr{B}}$. Notice that, by definition and (2.7), the inertia tensor $\mathbb{J}_O$ is time-independent with components in $\mathcal{R}$ given by

$$(\mathbb{J}_O)_{ij} = \int_{\mathcal{B}}\rho_{\mathcal{B}}(\boldsymbol{e}_i\times\boldsymbol{x})\cdot(\boldsymbol{e}_j\times\boldsymbol{x})\,,\quad i,j=1,2,3\,,$$

where $\rho_{\mathcal{B}}(x) = \rho_{\mathscr{B}}(\mathbb{Q}(t) \cdot \boldsymbol{x} + y_O)$.

We may thus conclude that in a body-fixed frame $\mathcal{R}$, the motion of a (possibly constrained) body with an interior cavity entirely filled with a liquid (viscous or inviscid) is governed by equations (2.15)–(2.19), in the unknowns $(\boldsymbol{u}, \widetilde{p}, \boldsymbol{\xi}_G, \boldsymbol{\omega})$.[1]

In several significant situations that we will encounter later on, we find it more convenient to write (2.17)–(2.19) in terms of the relative velocity $\boldsymbol{v}$ (see (2.14)). In this regard, from (2.17)–(2.19) we find

$$\left.\begin{aligned}&\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \dot{\boldsymbol{\omega}} \times \boldsymbol{x} + 2\boldsymbol{\omega} \times \boldsymbol{v}\Big) = \mu\, \Delta \boldsymbol{v} - \nabla \mathsf{p} \\ &\operatorname{div} \boldsymbol{v} = 0\end{aligned}\right\} \quad (x,t) \in \mathcal{C} \times (0,\infty)$$
$$\left.\begin{aligned}&\boldsymbol{v}(x,t) = \boldsymbol{0}\,, \quad (\text{if } \mu > 0) \\ &\boldsymbol{v}(x,t) \cdot \boldsymbol{n} = \boldsymbol{0}\,, \quad (\text{if } \mu = 0)\end{aligned}\right\} \quad (x,t) \in \partial\mathcal{C} \times (0,\infty) \tag{2.20}$$

and

$$\begin{aligned}&M\, \dot{\boldsymbol{\xi}}_G + M\, \boldsymbol{\omega} \times \boldsymbol{\xi}_G = \boldsymbol{F} + \boldsymbol{\Phi} \\ &\dot{\boldsymbol{K}}_O + \boldsymbol{\omega} \times \boldsymbol{K}_O + M\, \boldsymbol{\xi}_O \times \boldsymbol{\xi}_G = \boldsymbol{M}_O + \boldsymbol{\tau}_O\,,\end{aligned} \tag{2.21}$$

with $\mathsf{p} := \widetilde{p} + \rho\, (\boldsymbol{\omega} \times \boldsymbol{\xi}_O + \dot{\boldsymbol{\xi}}_O) \cdot \boldsymbol{x} - \frac{1}{2}\rho(\boldsymbol{\omega} \times \boldsymbol{x})^2$,

$$\boldsymbol{K}_O := \mathbb{I}_O \cdot \boldsymbol{\omega} + M\, \boldsymbol{x}_G \times \boldsymbol{\xi}_O + \int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \boldsymbol{v}\,, \tag{2.22}$$

and

$$(\mathbb{I}_O)_{ij} = (\mathbb{J}_O)_{ij} + \int_{\mathcal{C}} \rho\, (\boldsymbol{x} \times (\boldsymbol{e}_j \times \boldsymbol{x}))_i\,. \tag{2.23}$$

In view of the identity $[\boldsymbol{x} \times (\boldsymbol{e}_j \times \boldsymbol{x})]_i = (\boldsymbol{e}_i \times \boldsymbol{x}) \cdot (\boldsymbol{e}_j \times \boldsymbol{x})$, we recognize that $\mathbb{I}_O$ is the inertia tensor, with respect to O, of the *whole* system $\mathscr{S}$ viewed as a single rigid body. Also, in deriving (2.22) we have used the following relation

$$M_{\mathcal{B}} \boldsymbol{x}_C + \int_{\mathcal{C}} \rho\, \boldsymbol{x} = M\, \boldsymbol{x}_G\,. \tag{2.24}$$

Remark 2.2.1. The governing equations in the form (2.20)–(2.23) reveal an important feature. Actually, in the case of a freely moving body ($\boldsymbol{\Phi} \equiv \boldsymbol{\tau}_O \equiv \boldsymbol{0}$), if we pick $O \equiv G$ the motion of the center of mass (described by $(2.21)_1$) decouples from the other equations, so that we may first solve for (2.15)–(2.16) and $(2.21)_2$–(2.23), and successively derive the motion of the center of mass from $(2.21)_1$. If, in addition, $\mathscr{S}$ moves by *inertial motion*, namely, $\boldsymbol{\mathcal{F}} \equiv \boldsymbol{\mathcal{M}}_O \equiv \boldsymbol{0}$, also equations (2.15)–(2.16) decouple from the others, so that the motion of $\mathscr{S}$ may be reduced to the study of (2.20), $(2.21)_2$ and (2.22)–(2.23).

[1] Notice that, from the rigid-body formula, $\boldsymbol{\xi}_O = \boldsymbol{\xi}_G + \boldsymbol{\omega} \times (O - G)$.

Chapter 3

Review of Classical Results on the Motion of the Rigid Body in Absence of Liquid

The fundamental aspect that characterizes the motion of a rigid body with a liquid-filled interior cavity is that the presence of the liquid may dramatically change the dynamics of the body, often (but not always!) exerting a substantial stabilizing influence. In order to better describe such an important issue, we find it appropriate to collect and review a number of classical and relevant results concerning the dynamics of a rigid body under the action of a system of given external forces.

To begin with, we observe that from (2.18) it follows that, in absence of liquid, the equations of motion of the rigid body in the frame $\mathcal{R}$ reduce to the following classical ones

$$\begin{aligned} &M_{\mathcal{B}}\,\dot{\boldsymbol{\xi}}_C + M_{\mathcal{B}}\,\boldsymbol{\omega}\times\boldsymbol{\xi}_C = \boldsymbol{F}+\boldsymbol{\Phi} \\ &\mathbb{J}_O\cdot\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbb{J}_O\cdot\boldsymbol{\omega}) + M_{\mathcal{B}}\,\boldsymbol{\xi}_O\times\boldsymbol{\xi}_C = \boldsymbol{M}_O+\boldsymbol{\tau}_O\,, \end{aligned} \tag{3.1}$$

in conjunction with equations (2.15)–(2.16).

Particularly for the important role that it will play in our later considerations, we would like to recall the *basic properties of the inertia tensor* $\mathbb{J}_O$. Since it is symmetric, we can always find a body-fixed frame, $\{O,\boldsymbol{e}_i\}$, with $\{\boldsymbol{e}_i\}$ ortho-normalized eigenvectors of $\mathbb{J}_O$, where $\mathbb{J}_O$ becomes diagonal. This frame [resp. its axes] is referred to as *principal frame* [resp. *axes*] *of inertia* and, if $O\equiv C$, as *central frame* [resp. *axes*] *of inertia*. The only non-zero components of $\mathbb{J}_O$ in $\{O,\boldsymbol{e}_i\}$ are denoted by $\mathsf{A}\,(\equiv(\mathbb{J}_O)_{11})$, $\mathsf{B}\,(\equiv(\mathbb{J}_O)_{22})$, and $\mathsf{C}\,(\equiv(\mathbb{J}_O)_{33})$, and called *principal* [resp. *central*] *moments of inertia*. Mathematically, they are eigenvalues corresponding to the eigenvectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, respetively. From the physical viewpoint, they represent the moments of inertia of $\mathscr{B}$ with respect to, in the order, $(O,\boldsymbol{e}_1)$, $(O,\boldsymbol{e}_2)$, and $(O, \boldsymbol{e}_3)$. Precisely,

$$\mathsf{A} = \int_{\mathcal{B}} \widehat{\rho}\,(x_2^2 + x_3^2)\,, \quad \mathsf{B} = \int_{\mathcal{B}} \widehat{\rho}\,(x_1^2 + x_3^2)\,, \quad \mathsf{C} = \int_{\mathcal{B}} \widehat{\rho}\,(x_1^2 + x_2^2)\,, \tag{3.2}$$

where $\widehat{\rho} = \widehat{\rho}(x)$ is the (mass) density of $\mathscr{B}$.

In the present chapter we shall present and analyze a number of results pertaining to solutions to the system (3.1), under different assumptions on the external forces. One of our main objectives will be the study of the stability properties, in the sense of Lyapunov, of steady-state (time-independent) solutions to $(3.1)_2$–(2.15)–(2.16). In this regard, we shall frequently use the classical stability and instability criteria of Lyapunov and Chetayev that, for reader's sake, we would like to recall in their general formulation.

Consider the differential equation

$$\dot{\boldsymbol{X}} = \boldsymbol{f}(\boldsymbol{X})\,, \quad \boldsymbol{X} \in \mathbb{R}^N \tag{3.3}$$

where $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$ with $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$ is enough smooth as to ensure existence, uniqueness and continuous data-dependence of solutions to (3.3). We shall say that the equilibrium $\boldsymbol{X} = \boldsymbol{0}$ is *stable*, if for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$|\boldsymbol{X}(0)| < \delta \ \text{ implies } \ \sup_{t\geq 0} |\boldsymbol{X}(t)| < \varepsilon\,,$$

and *unstable* otherwise. The following classical results hold (e.g., [5, Theorems 1.1 and 1.2 in Chapter X].

Proposition 3.0.1. *Suppose there is $R > 0$ and a continuous function $V : \boldsymbol{X} \in B_R(\boldsymbol{0}) \mapsto V(\boldsymbol{X}) \in [0, \infty)$ such that* (i) $V(\boldsymbol{0}) = 0$ *and* $V(\boldsymbol{X}) > 0$ *for* $\boldsymbol{X} \neq \boldsymbol{0}$, *and* (ii) $\sup_{t\geq 0} V(\boldsymbol{X}(t)) \leq V(\boldsymbol{X}(0))$, *whenever V is evaluated along the solutions to* (3.3). *Then, $\boldsymbol{X} = \boldsymbol{0}$ is stable.*

Proposition 3.0.2. *Suppose there exist a neighborhood U of $\boldsymbol{X} = \boldsymbol{0}$, an open set U_1 with $\boldsymbol{0} \in \overline{U}_1$, and and a function $V : U_1 \mapsto [0, \infty)$ of class C^1 with the following properties:* (i) $V(\boldsymbol{X}) = 0$ *at all* $\boldsymbol{X} \in \partial U_1 \cap U$, *and* (ii) $V(\boldsymbol{0}) = 0$, *and* $V, \dot{V} > 0$ *in* $U_1 \cap U - \{\boldsymbol{0}\}$, *the derivative being evaluated along the solutions to* (3.3). *Then $\boldsymbol{X} = \boldsymbol{0}$ is unstable.*

3.1 Inertial Motions

We begin to consider the simplest situation, namely, when $\mathscr{B}$ moves freely in the whole space, in absence of external forces, namely, $\boldsymbol{F}(t) = \boldsymbol{M}_O(t) = \boldsymbol{\Phi} = \boldsymbol{\tau}_O \equiv \boldsymbol{0}$. Corresponding motions are often referred to as *inertial motions*. As we already mentioned (see Remark 2.2.1), in such a case (3.1) and (2.15)–(2.16) decouple. In what follows, we will focus our analysis on the properties of the solutions to

(3.1).[1] From $(3.1)_1$ we then obtain $\dot{\boldsymbol{\xi}}_C + \boldsymbol{\omega} \times \boldsymbol{\xi} = \mathbf{0}$, which means that the center of mass C moves by uniform and rectilinear motion in the inertial frame $\mathcal{I}$. Thus, choosing $O' \equiv O \equiv C$, the dynamics of $\mathscr{B}$ reduces to study its *motion around* C, which means to find all possible solutions to the Euler equation

$$\mathbb{J}_C \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbb{J}_C \cdot \boldsymbol{\omega}) = \mathbf{0}. \tag{3.4}$$

This equation admits two fundamental *first integrals*. In fact, dot-multiplying (3.4) by $\boldsymbol{\omega}$ we at once deduce that the kinetic energy, T, must be conserved along the generic solution:

$$T(t) := \tfrac{1}{2}\boldsymbol{\omega}(t) \cdot \mathbb{J}_C \cdot \boldsymbol{\omega}(t) = \text{const.} \tag{3.5}$$

Likewise, dot-multiplying (3.4) by $\mathbb{J}_C \cdot \boldsymbol{\omega}$ we recover that the magnitude of the angular momentum must be conserved as well:

$$|\mathbb{J}_C \cdot \boldsymbol{\omega}(t)|^2 = \text{const.} \tag{3.6}$$

We next observe that, choosing $\{C, \boldsymbol{e}_i\}$ coinciding with the central frame of inertia, we obtain

$$\mathbb{J}_C \cdot \boldsymbol{\omega} = \mathsf{A}\,\omega_1\,\boldsymbol{e}_1 + \mathsf{B}\,\omega_2\,\boldsymbol{e}_2 + \mathsf{C}\,\omega_3\,\boldsymbol{e}_3\,, \tag{3.7}$$

with A, B and C are central moments of inertia given in (3.2). We can then write equation (3.4) component-wise as follows

$$\begin{aligned} &\mathsf{A}\,\dot{\omega}_1 - (\mathsf{B} - \mathsf{C})\,\omega_2\,\omega_3 = 0 \\ &\mathsf{B}\,\dot{\omega}_2 - (\mathsf{C} - \mathsf{A})\,\omega_3\,\omega_1 = 0 \\ &\mathsf{C}\,\dot{\omega}_3 - (\mathsf{A} - \mathsf{B})\,\omega_1\,\omega_2 = 0\,. \end{aligned} \tag{3.8}$$

In the next subsections we shall analyze some important properties of relevant solutions to (3.8). To this end, and without loss of generality, we will assume throughout

$$\mathsf{A} \le \mathsf{B} \le \mathsf{C}\,.$$

3.1.1 Steady-State Motions and their Stability Properties

These motions are characterized by the condition $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and are often referred to as *permanent rotations*. From (3.8) we obtain at once the well-known result that *permanent rotations may occur only around the central axes of inertia*, whose direction is determined, for a fixed body shape, solely by the distribution of mass of $\mathscr{B}$. In view of (3.4), we obviously derive that in a permanent rotation the axis of rotation is parallel to the direction of the angular momentum, which is constant in the inertial frame $\mathcal{I}$. We also observe that, due to the uniqueness theorem for the initial-value problem associated to system (3.8), permanent rotations may

[1] As a matter of fact, once (3.1) are solved, namely, the angular velocity is found, the function $\mathbb{Q}(t)$ and hence the motion of $\mathscr{B}$ in $\mathcal{I}$, is obtained via a classical procedure; see [10, §3.4].

effectively be realized by initially imparting a rotation around any of the central axes.

We wish now to investigate the stability properties of such motions with the help of Proposition 3.0.1 and Proposition 3.0.2. In this regard, denote by $\mathsf{s}_0 := \omega_0\, \boldsymbol{e}$, $\omega_0 \in \mathbb{R} - \{0\}$, $\boldsymbol{e} \in \{\boldsymbol{e}_i\}$, a generic permanent rotation.[2] As originally proved in [1, §15], these properties can be fully characterized in terms of the moment of inertia with respect to that central axis around which they occur; see Theorem 3.1.1.

Thus, let $\mathsf{s}_0 := p_0\, \boldsymbol{e}_1$ be the basic motion, and let

$$\boldsymbol{\omega}(t) := p_0\, \boldsymbol{e}_1 + \boldsymbol{\zeta}(t)\,,\ \omega_2 = \zeta_2(t)\,,\ \omega_3 = \zeta_3(t)$$

be a corresponding "perturbed motion." From (3.5) and (3.6) we thus get

$$2T(t) := \mathsf{A}\, \zeta_1^2(t) + \mathsf{B}\, \zeta_2^2(t) + \mathsf{C}\, \zeta_3^2(t) + 2\mathsf{A}\, p_0 \zeta_1(t) = \text{const.} \tag{3.9}$$

and

$$K(t) := \mathsf{A}^2\, \zeta_1^2(t) + \mathsf{B}^2\, \zeta_2^2(t) + \mathsf{C}^2\, \zeta_3^2(t) + 2\mathsf{A}^2\, p_0\, \zeta_1(t) = \text{const.}\,. \tag{3.10}$$

Define

$$V(\boldsymbol{\zeta}) := (K - 2\mathsf{A}\, T) + 4T^2\,. \tag{3.11}$$

By direct substitution, from (3.9) and (3.10) we find

$$V(\boldsymbol{\zeta}) = \mathsf{B}\,(\mathsf{B}-\mathsf{A})\zeta_2^2 + \mathsf{C}\,(\mathsf{C}-\mathsf{A})\zeta_3^2 + (\mathsf{A}\, \zeta_1^2 + 2\mathsf{A}\, \zeta_1 p_0 + \mathsf{B}\, \zeta_2^2 + \mathsf{C}\, \zeta_3^2)^2$$

which shows that if

$$\mathsf{A} < \mathsf{B} \le \mathsf{C}, \tag{3.12}$$

then $V(\boldsymbol{\zeta}) \ge 0$. Actually, $V(\boldsymbol{\zeta}) = 0$ if and only if $\boldsymbol{\zeta} = \mathbf{0}$, provided $\boldsymbol{\zeta}$ ranges in a suitable ball centered at $\mathbf{0}$. In fact, if $V(\boldsymbol{\zeta}) = 0$, from (3.12) we deduce $\zeta_2 = \zeta_3 = 0$ and either $\zeta_1 = 0$ or else $\zeta_1 = -2p_0$. However, the latter is excluded if, for example, $\boldsymbol{\zeta} \in B_{|p_0|}(0)$. Thus, V is positive definite in $B_{|p_0|}(0)$. Moreover, from (3.9)–(3.11) we also have $\dot V = 0$, so that, by Proposition 3.0.1, we conclude that under the assumption (3.12), namely, the axis of rotation is the one of minimum moment of inertia, the permanent rotation $p_0\, \boldsymbol{e}_1$ is stable. Likewise, if $\mathsf{s}_0 = r_0\, \boldsymbol{e}_3$ the two first integrals become

$$\begin{aligned} 2T(t) &:= \mathsf{A}\, \zeta_1^2(t) + \mathsf{B}\, \zeta_2^2(t) + \mathsf{C}\, \zeta_3^2(t) + 2\mathsf{C}\, r_0 \zeta_3(t) = \text{const.} \\ K(t) &:= \mathsf{A}^2\, \zeta_1^2(t) + \mathsf{B}^2\, \zeta_2^2(t) + \mathsf{C}^2\, \zeta_3^2(t) + 2\mathsf{C}^2\, r_0\, \zeta_3(t) = \text{const.}\,. \end{aligned}$$

Thus, by choosing this time

$$V(\boldsymbol{\zeta}) := (2\mathsf{C}\, T - K) + 4T^2\,. \tag{3.13}$$

[2] Observe that for any initial datum $\boldsymbol{\omega}_0 \in \mathbb{R}^3$, the corresponding initial-value problem associated to (3.8) has a unique, global and smooth solution that equals $\boldsymbol{\omega}_0$ at time $t = 0$. This is consequence of classical results on system of ODE's along with the uniform bound $|\boldsymbol{\omega}(t)| \le \text{const.}$ that follows from conservation of energy (3.5).

and employing an entirely similar argument, one can show that if

$$\mathsf{A} \leq \mathsf{B} < \mathsf{C}\,, \tag{3.14}$$

the permanent rotation $r_0\,\boldsymbol{e}_3$ (occurring around the axis of maximum moment of inertia) is stable.

It remains to investigate the case

$$\mathsf{A} < \mathsf{B} < \mathsf{C}\,, \tag{3.15}$$

and $\mathsf{s}_0 := q_0\,\boldsymbol{e}_2$, namely, the permanent rotation occurs around the axis of intermediate moment of inertia. We shall show that this rotation is unstable. Actually, choosing

$$V = -\zeta_1\zeta_3$$

from (3.8) we deduce

$$\dot{V} = (\zeta_2 + q_0)\,\frac{\mathsf{C}\,(\mathsf{C}-\mathsf{B})\,\zeta_3^3 + \mathsf{A}\,(\mathsf{B}-\mathsf{A})\,\zeta_1^2}{\mathsf{A}\,\mathsf{C}}\,.$$

Therefore, defining

$$U = \{\boldsymbol{\zeta} \in \mathbb{R}^3 :\ \zeta_2 > -q_0\}\,,\quad U_1 = \{\boldsymbol{\zeta} \in \mathbb{R}^3 :\ \zeta_1\,\zeta_3 < 0\}\,,$$

we immediately recognize that all assumptions of Proposition 3.0.2 are satisfied, which implies that $\mathsf{s}_0 := q_0\,\boldsymbol{e}_2$ is unstable.

The above results are summarized in the following.

Theorem 3.1.1. (Stability of Permanent Rotations) *Permanent rotations occurring around the central axes of either maximum or minimum moment of inertia are stable. Those around the intermediate central axis are, instead, unstable.*

3.1.2 Unsteady Motions.

As is well known, the simplest situation occurs when two of the central moments of inertia coincide. In this case, equations (3.8) admit a class of remarkable particular solutions known as *regular precessions*. In fact assume, to fix the ideas, $\mathsf{A} = \mathsf{B}$, so that from (3.8) we deduce for some $r_0 \in \mathbb{R}$,

$$\begin{aligned}
&\mathsf{A}\,\dot{\omega}_1 - (\mathsf{A}-\mathsf{C})\,r_0\,\omega_2 = 0\\
&\mathsf{A}\,\dot{\omega}_2 + (\mathsf{A}-\mathsf{C})\,r_0\,\omega_1 = 0\\
&\omega_3(t) = r_0\,,\quad \text{all } t \geq 0\,,
\end{aligned}$$

which implies

$$\boldsymbol{\omega}(t) = \omega_1(t)\,\boldsymbol{e}_1 + \omega_2(t)\,\boldsymbol{e}_2 + r_0\,\boldsymbol{e}_3\,, \tag{3.16}$$

where

$$\begin{aligned}\omega_1(t) &= \omega_1(0)\cos(\Omega\, t) + \omega_2(0)\sin(\Omega\, t)\,,\\ \omega_2(t) &= \omega_2(0)\cos(\Omega\, t) - \omega_1(0)\sin(\Omega\, t)\,,\end{aligned} \tag{3.17}$$

and $\Omega := (1 - \mathsf{C}/\mathsf{A})r_0$, $\omega_1(0) \equiv -\dot{\omega}_2(0)/\Omega$, $\omega_2(0) \equiv \dot{\omega}_1(0)/\Omega$. Notice that, in view of the uniqueness property for the initial-value problem associated to (3.8), the motion described by (3.16)–(3.17) can effectively be realized by prescribing appropriate initial conditions.

Now, from (3.16) and (3.7) we also obtain

$$\mathbb{J}_C \cdot \boldsymbol{\omega} = \mathsf{A}\,(\omega_1\,\boldsymbol{e}_1 + \omega_2\,\boldsymbol{e}_2) + \mathsf{C}\,r_0\,\boldsymbol{e}_3\,,$$

which in combination with (3.16) provides

$$\boldsymbol{\omega}(t) = \frac{1}{\mathsf{A}}\,\mathbb{J}_C \cdot \boldsymbol{\omega} + \left(1 - \frac{\mathsf{C}}{\mathsf{A}}\right) r_0 \boldsymbol{e}_3\,.$$

Therefore, we find that the angular velocity is the sum of two vectors, the first ($\mathbb{J}_C \cdot \boldsymbol{\omega}$) being constant in the inertial frame, while the other ($(1 - \mathsf{C}/\mathsf{A})r_0\,\boldsymbol{e}_3$) constant in the body-fixed frame. As a result, by the theory of compound motions [21, pp. 165–167], this allows us to conclude that the motion of $\mathscr{B}$ is a regular precession. Precisely, when observed from $\mathcal{I}$, $\mathscr{B}$ rotates uniformly around $\boldsymbol{e}_3$ while $\boldsymbol{e}_3$ rotates, also uniformly, around the constant direction of the angular momentum, forming with the latter the constant (nutation) angle $\theta = \cos^{-1}(\mathsf{C}\,r_0/|\mathbb{J}_C \cdot \boldsymbol{\omega}|)$.

In absence of "mass symmetry", namely, $\mathsf{A} < \mathsf{B} < \mathsf{C}$, the resolution of system (3.8) under given initial condition becomes much more involved, mainly, due to the presence of the nonlinear terms. As a consequence, the motion of $\mathscr{B}$ appears to be somehow complicated to resolve. However, as shown originally by Jacobi [7] one can still obtain an analytical closed-form solution in terms of (Jacobi) elliptic functions:

$$\omega_1(t) = a_1\,\mathrm{cn}\,(\sigma\, t)\,, \quad \omega_2(t) = a_2\,\mathrm{sn}\,(\sigma\, t)\,, \quad \omega_3(t) = a_3\,\mathrm{dn}\,(\sigma\, t)\,,$$

where a_i, $i = 1, 2, 3$, are constants and $\sigma = \sigma(\mathsf{A}, \mathsf{B}, \mathsf{C})$; see [10, §3].

Alternatively, the motion of $\mathscr{B}$ admits the classical and elegant geometric representation due to Poinsot [15]. More precisely, $\mathscr{B}$ moves as if it were rigidly connected to the ellipsoid of inertia

$$\{(x, y, z) \in \mathbb{R}^3 : \ \mathsf{A}\,x^2 + \mathsf{B}\,y^2 + \mathsf{C}\,z^2 = 1\}$$

and this, in turn, as a rigid body, would roll without sliding on a fixed plane orthogonal to the (constant) direction of the angular momentum; see, for details [10, §4].

3.2 Heavy Body

We now consider the case when $\mathscr{B}$ moves under the action of gravity, $\boldsymbol{g}$, *that we assume to have direction and orientation of the axis* $-\boldsymbol{e}_3'$ *in the inertial frame* $\mathcal{I}$.

If the body is entirely unconstrained, then its dynamics can be reduced to cases that we already analyzed in previous subsections. Actually, the torque due to the weight $\boldsymbol{p} = M_{\mathcal{B}}\, \boldsymbol{g}$ of $\mathscr{B}$ with respect to its center of mass C is identically vanishing. As a consequence, from (3.1) with $O \equiv C$ we get that, on the one hand, the motion of $\mathscr{B}$ around C is governed by (3.4) –a case we already discussed– and, on the other hand, C moves, with respect to the inertial frame, like a point-mass with mass $M_{\mathcal{B}}$, subject to the constant force $\boldsymbol{p}$, which represents a classical and elementary problem in Newtonian particle mechanics.

Therefore, in order to make its dynamics more interesting, $\mathscr{B}$ must be suitably constrained. Two most significant and classical examples in this regard, are when $\mathscr{B}$ moves by keeping fixed (with respect to the inertial frame) either *a point*, P (say) *rigidly fixed to it*, that is, $\mathscr{B}$ executes a *spherical motion* around P, or else *all points of a straight line* r passing through $\mathscr{B}$, namely, $\mathscr{B}$ executes a *rotational motion* around r.

Objective of the following subsections is to review the relevant features of these two classes of motions and to present corresponding main results.

3.2.1 Motions Around a Fixed Point

Let us choose $O \equiv P$ and suppose that *the constraint is frictionless*. Under this assumption, one can readily show that $\boldsymbol{\tau}_O = \boldsymbol{0}$. Moreover, setting $\mathsf{g} := \boldsymbol{g}/g$, we have

$$\boldsymbol{M}_O(t) = \mathbb{Q}^\top(t) \cdot [(y_C - y_O) \times \boldsymbol{p}] = M_{\mathcal{B}}\, g\, \boldsymbol{x}_C \times (\mathbb{Q}^\top(t) \cdot \mathsf{g})\,.$$

As a result, if we define $\boldsymbol{\gamma} := \mathbb{Q}^\top(t) \cdot \mathsf{g}$, the system (3.1) becomes

$$\begin{aligned} M_{\mathcal{B}}\, \dot{\boldsymbol{\xi}}_C + M_{\mathcal{B}}\, \boldsymbol{\omega} \times \boldsymbol{\xi}_C &= M_{\mathcal{B}}\, g\, \boldsymbol{\gamma} + \boldsymbol{\Phi} \\ \mathbb{J}_O \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbb{J}_O \cdot \boldsymbol{\omega}) &= M_{\mathcal{B}}\, g\, \boldsymbol{x}_C \times \boldsymbol{\gamma}\,, \end{aligned} \tag{3.18}$$

to which we have to add the equation for the unknown function $\boldsymbol{\gamma} = \boldsymbol{\gamma}(t)$ that, by (2.15)–(2.16), takes the form of the classical Poisson equation:

$$\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = \boldsymbol{0}\,. \tag{3.19}$$

At this point we observe that equations $(3.18)_2$–(3.19) and $(3.18)_1$ decouple. In fact, once we solve $(3.18)_2$–(3.19) for $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ we can then obtain the motion of the center of mass C from $(3.18)_1$. In what follows we will focus our attention on solutions to $(3.18)_2$–(3.19), commenting about the motion of C as necessary.

As in the case of inertial motions, also in the current situation we can show that $(3.18)_2$–(3.19) admits two relevant first integrals. More precisely, dot-multiplying both sides of $(3.18)_2$ by $\boldsymbol{\omega}$ and those of (3.19) by $\boldsymbol{x}_C$ we deduce the *conservation of total energy*:

$$T(t) - U(t) := \tfrac{1}{2}\,\boldsymbol{\omega}(t)\cdot\mathbb{J}_O\cdot\boldsymbol{\omega}(t) - M_{\mathcal{B}}g\,\boldsymbol{\gamma}(t)\cdot\boldsymbol{x}_C = \text{const.} \tag{3.20}$$

We next dot-multiply $(3.18)_2$ by $\boldsymbol{\gamma}$, (3.19) by $\mathbb{J}_O\cdot\boldsymbol{\omega}$ and sum side-by-side the resulting equations. This produces the *conservation of the vertical component of the total angular momentum*:

$$\boldsymbol{\gamma}(t)\cdot\mathbb{J}_O\cdot\boldsymbol{\omega}(t) = \text{const.} \tag{3.21}$$

Finally, to the above integrals, we should add a third one, of entirely kinematic nature, obtained by dot-multiplying both sides of (3.19) by $\boldsymbol{\gamma}$:

$$\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) = 1\,. \tag{3.22}$$

The integrals (3.20)–(3.22) will play a fundamental role in the stability analysis that we shall perform later on.

3.2.1.1 Steady-State Motions.

This type of motion is defined as one where the angular velocity is constant in time, $\dot{\boldsymbol{\omega}} = \mathbf{0}$; in other words, $\mathscr{B}$ executes a *permanent rotation*. As expected, a permanent rotation may occur if and only if $\dot{\boldsymbol{\gamma}} = \mathbf{0}$ as well. To see this, we observe that from $(3.18)_2$ with $\boldsymbol{\omega}(t) = \mathbf{const.}$ it follows $\boldsymbol{x}_C\times\boldsymbol{\gamma}(t) = \mathbf{const.}$ Dot-multiplying both sides of (3.19) by $\boldsymbol{x}_C$ gives

$$\overset{\cdot}{\overbrace{\boldsymbol{\gamma}\cdot\boldsymbol{x}_C}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}\cdot\boldsymbol{x}_C\,,$$

whereas dot-multiplying by $\boldsymbol{\omega}$ those of $(3.18)_2$ (with $\dot{\boldsymbol{\omega}} = \mathbf{0}$) furnishes $\boldsymbol{\gamma}\times\boldsymbol{\omega}\cdot\boldsymbol{x}_C = 0$, and we conclude $\boldsymbol{\gamma} = \mathbf{const.}$ Conversely, if $\boldsymbol{\gamma}(t) = \mathbf{const.}$ then, from (3.19) it follows $\boldsymbol{\omega} = \lambda(t)\,\boldsymbol{\gamma}$. However, dot-multiplying both sides of $(3.18)_2$ by $\boldsymbol{\omega}$ and using the latter, we conclude $\lambda(t) = \text{const}$. Consequently, *permanent rotations may occur if and only if both* $\boldsymbol{\omega}$ *and* $\boldsymbol{\gamma}$ *are constant.* In such a case, using (3.18)–(3.19), it follows that $(\boldsymbol{\omega},\boldsymbol{\gamma})$ must satisfy the following system

$$\boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}\,,\quad \boldsymbol{\omega}\times(\mathbb{J}_O\cdot\boldsymbol{\omega}) = M_{\mathcal{B}}\,g\,\boldsymbol{x}_C\times\boldsymbol{\gamma}\,. \tag{3.23}$$

The first of these equations furnishes $\boldsymbol{\omega} = \lambda\,\boldsymbol{\gamma}$ for some $\lambda\in\mathbb{R}$. Moreover, if we choose the body-fixed frame $\mathcal{R}$ as principal frame of inertia with respect to O, we get

$$\mathbb{J}_O\cdot\boldsymbol{\omega} = \mathsf{A}\,\omega_1\,\boldsymbol{e}_1 + \mathsf{B}\,\omega_2\,\boldsymbol{e}_2 + \mathsf{C}\,\omega_3\,\boldsymbol{e}_3\,,$$

with A, B and C principal moments of inertia. As a consequence, denoting by (x_0, y_0, z_0) the coordinates of $\boldsymbol{x}_C$, the equations in (3.23) are equivalent to the following ones in the unknowns λ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$

$$\left\{\begin{aligned} &\lambda^2(\mathsf{C}-\mathsf{B})\gamma_2\gamma_3 = M_{\mathcal{B}}\, g\,(y_0\gamma_3 - z_0\gamma_2),\\ &\lambda^2(\mathsf{A}-\mathsf{C})\gamma_1\gamma_3 = M_{\mathcal{B}}\, g\,(z_0\gamma_1 - x_0\gamma_3),\\ &\lambda^2(\mathsf{B}-\mathsf{A})\gamma_1\gamma_2 = M_{\mathcal{B}}\, g\,(x_0\gamma_2 - y_0\gamma_1),\\ &\boldsymbol{\omega} = \lambda\,\boldsymbol{\gamma}\,,\quad \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1. \end{aligned}\right. \tag{3.24}$$

From the physical viewpoint, this means that, when viewed from the inertial frame, in a permanent rotation the body uniformly rotates around the vertical axis passing through O, a_O, with an "inclination" with respect to a_O characterized by direction cosines γ_1, γ_2, and γ_3 that solve (3.24) along with a corresponding angular velocity, for a given mass of the body and location of its center of mass.

System (3.24) admits, in particular, the following set of elementary but noteworthy solutions:

$$\boldsymbol{\gamma} \equiv \pm\,\boldsymbol{e}_i\,,\ i = 1, 2, 3,$$

with $\boldsymbol{\omega}$ given in $(3.24)_4$ for some $\lambda \neq 0$. They represent permanent rotations occurring around one of the principal axes, oriented along the direction of gravity. However, in general, the class of solutions to (3.21) is extremely rich. A complete and detailed analysis is performed in [10, §§8.31, 8.32].

For the applications we have in mind, here we shall limit ourselves to single out and discuss in details a special class of solutions corresponding to the case when the center of mass lies on one of the principal axes. To this purpose, we take $C \in (O, \boldsymbol{e}_3)$ and orient the frame $\{O, \boldsymbol{e}_i\}$ in such a way that $z_0 > 0$. Thus, setting

$$\beta^2 := M_{\mathcal{B}}\, g\, z_0\ \ (>0)\,, \tag{3.25}$$

equations (3.24) specialize to the following ones

$$\left\{\begin{aligned} &\lambda^2(\mathsf{C}-\mathsf{B})\gamma_2\gamma_3 = -\beta^2\,\gamma_2,\\ &\lambda^2(\mathsf{A}-\mathsf{C})\gamma_1\gamma_3 = \beta^2\,\gamma_1,\\ &\lambda^2(\mathsf{B}-\mathsf{A})\gamma_1\gamma_2 = 0\,,\\ &\boldsymbol{\omega} = \lambda\,\boldsymbol{\gamma}\,,\quad \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1. \end{aligned}\right. \tag{3.26}$$

In order to characterize and describe the physical meaning of the class, S, of solutions to this system, we begin to observe that from (3.25) and (3.26) it follows that every element of S must have $\gamma_3 \neq 0$, which means that in every steady-state motion the $\boldsymbol{e}_3$-axis cannot be horizontal, namely, either $\boldsymbol{\gamma}\|\boldsymbol{e}_3$, or $|\gamma_3| \in (0,1)$. Moreover, again from (3.26), we find that $\lambda = 0$ (no motion) is possible if and only if $\boldsymbol{\gamma} = \pm\,\boldsymbol{e}_3$. We then conclude that the class S satisfies

$$\mathrm{S} \supset \mathrm{PR}\,, \tag{3.27}$$

where

$$\mathrm{PR} := \{(\boldsymbol{\omega},\boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2 : \ \boldsymbol{\gamma} = \pm\, \boldsymbol{e}_3,\ \boldsymbol{\omega} = \lambda\boldsymbol{\gamma}\,,\ \text{some } \lambda \in \mathbb{R}\}\,. \tag{3.28}$$

Clearly, members of PR are “pure” rotations, where $\mathscr{B}$ rotates with constant (possibly zero) angular velocity around $\boldsymbol{e}_3$, with $\boldsymbol{e}_3$ parallel to a_O. Our next task is to find all other motions in S that are not pure rotations. Setting $\mathrm{PR}' := \mathrm{S} - \mathrm{PR}$, by what we said we have

$$\mathrm{PR}' \subset \{(\boldsymbol{\omega},\boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2 : \ \boldsymbol{\omega} = \lambda\boldsymbol{\gamma},\ \text{some } \lambda \in \mathbb{R} - \{0\},\ |\gamma_3| < 1\,,\ \gamma_3 \neq 0\}\,.$$

The characterization of PR' depends on the relative magnitude of the principal moments of inertia, according to the following cases.

(i) $\mathsf{A} = \mathsf{B} = \mathsf{C}$. From $(3.26)_{1,2,3}$ we obtain $\gamma_1 = \gamma_2 = 0$, $\gamma_3^2 = 1$, which implies

$$\mathrm{PR}' = \emptyset\,.$$

(ii) $\mathsf{A} = \mathsf{B} \neq \mathsf{C}$. From $(3.26)_3$ we find no further restrictions on γ_1, γ_2 other than that imposed by $\boldsymbol{\gamma} \in S^2$, and we deduce

$$\begin{aligned} \mathrm{PR}' \equiv \mathrm{SP}_0 := \{(\boldsymbol{\omega},\boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2 : \ &\boldsymbol{\omega} = \lambda\boldsymbol{\gamma},\ \lambda \in \mathbb{R} - \{0\}\,, \\ &\gamma_3 = -\frac{\beta^2}{\lambda^2(\mathsf{C} - \mathsf{A})}\,,\ |\gamma_3| < 1\}\,. \end{aligned} \tag{3.29}$$

(iii) $\mathsf{A} \neq \mathsf{B} \neq \mathsf{C}$. From $(3.26)_3$ we get either $\gamma_1 = 0$ or $\gamma_2 = 0$, and thus infer

$$\mathrm{PR}' = \mathrm{SP}_1 \cup \mathrm{SP}_2$$

where

$$\begin{aligned} \mathrm{SP}_1 := \{(\boldsymbol{\omega},\boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2 : \ &\boldsymbol{\omega} = \lambda\boldsymbol{\gamma},\ \lambda \in \mathbb{R} - \{0\}\,, \\ &\gamma_1 = 0\,,\ \gamma_3 = -\frac{\beta^2}{\lambda^2(\mathsf{C} - \mathsf{B})}\,,\ |\gamma_3| < 1\}\,. \\ \mathrm{SP}_2 := \{(\boldsymbol{\omega},\boldsymbol{\gamma}) \in \mathbb{R}^3 \times S^2 : \ &\boldsymbol{\omega} = \lambda\boldsymbol{\gamma},\ \lambda \in \mathbb{R} - \{0\}\,, \\ &\gamma_2 = 0\,,\ \gamma_3 = -\frac{\beta^2}{\lambda^2(\mathsf{C} - \mathsf{A})}\,,\ |\gamma_3| < 1\}\,. \end{aligned} \tag{3.30}$$

(iv) $\mathsf{A} \neq \mathsf{B} = \mathsf{C}$. From $(3.26)_1$ it follows $\gamma_2 = 0$, and so

$$\mathrm{PR}' = \mathrm{SP}_2\,.$$

(v) $\mathsf{C} = \mathsf{A} \neq \mathsf{B}$. From $(3.26)_2$ it follows $\gamma_1 = 0$, and so

$$\mathrm{PR}' = \mathrm{SP}_1\,.$$

All the above leads to the following characterization of the solution set S.

Theorem 3.2.1. - *Let* S *be the class of solutions to* (3.26). *Then, the following holds.*

1. *If* $\mathsf{A} = \mathsf{B} = \mathsf{C}$, *then* $\mathrm{S} = \mathrm{PR}$*;*
2. *If* $\mathsf{A} = \mathsf{B} \neq \mathsf{C}$, *then* $\mathrm{S} = \mathrm{PR} \cup \mathrm{SP}_0$*;*
3. *If* $\mathsf{A} \neq \mathsf{B} \neq \mathsf{C}$, *then* $\mathrm{S} = \mathrm{PR} \cup \mathrm{SP}_1 \cup \mathrm{SP}_2$*;*
4. *If* $\mathsf{A} \neq \mathsf{B} = \mathsf{C}$, *then* $\mathrm{S} = \mathrm{PR} \cup \mathrm{SP}_2$*;*
5. *If* $\mathsf{C} = \mathsf{A} \neq \mathsf{B}$, *then* $\mathrm{S} = \mathrm{PR} \cup \mathrm{SP}_1$,

where PR *and* SP_i, $i = 0, 1, 2$ *are defined in* (3.28)–(3.30).

As for the physical meaning of solutions in the the classes SP_i, we see that they are representative of so-called *steady precessional* motions where, when observed from the inertial frame, $\mathscr{B}$ rotates uniformly around a_O, with the $\boldsymbol{e}_3$-axis describing a cone of constant semi-aperture

$$\cos^{-1}\left(-\frac{\beta^2}{\lambda^2(\mathsf{C}-\mathsf{A})}\right) \quad \text{for classes } \mathrm{SP}_i,\ i=0,2$$
$$\cos^{-1}\left(-\frac{\beta^2}{\lambda^2(\mathsf{C}-\mathsf{B})}\right) \quad \text{for class } \mathrm{SP}_1\,.$$

Moreover, the direction of the axis of rotation, which is parallel to the gravity, lies in the plane $(O; \boldsymbol{e}_2, \boldsymbol{e}_3)$ (for SP_1) and in the plane $(O; \boldsymbol{e}_1, \boldsymbol{e}_3)$ (for SP_1) We finally notice that for a steady precession to be effectively realized it is necessary (and sufficient) that the magnitude of the angular velocity $\boldsymbol{\omega}$ is sufficiently large and, precisely,

$$\omega^2 > \frac{\beta^2}{|\mathsf{C}-\mathsf{A}|} \quad \text{for classes } \mathrm{SP}_i,\ i=0,2$$
$$\omega^2 > \frac{\beta^2}{|\mathsf{C}-\mathsf{B}|} \quad \text{for class } \mathrm{SP}_1\,.$$

3.2.1.2 On the Stability of Steady-State Motions.

In this subsection, we will analyze the stability properties of some relevant solutions to (3.26). To this end, let

$$\mathsf{s}_0 := (\lambda\,\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_0) \in \mathbb{R}^3 \times S^2$$

be a *given* solution to (3.26), and let

$$\boldsymbol{\omega}(t) = \lambda\,\boldsymbol{\gamma}_0 + \boldsymbol{\zeta}(t)\,, \quad \boldsymbol{\gamma}(t) = \boldsymbol{\gamma}_0 + \boldsymbol{z}(t) \tag{3.31}$$

be a corresponding "perturbed motion", namely, a solution to $(3.18)_2$–(3.19), with $C \in (O, \boldsymbol{e}_3)$. As in the previous section, also here we shall use the methods of Lyapunov to investigate the stability of s_0. To this end, we observe that from the first integrals (3.20)–(3.22) it results that the following functions V_i, $i = 1, 2, 3$, are constant along solutions to $(3.18)_2$–(3.19):

$$\begin{aligned} V_1 &:= \mathsf{A}\,(\zeta_1^2 + 2\lambda\,\gamma_{01}\zeta_1) + \mathsf{B}\,(\zeta_2^2 + +2\lambda\,\gamma_{02}\zeta_2) + \mathsf{C}\,(\zeta_3^2 + 2\lambda\gamma_{03}\zeta_3) - 2\beta^2\, z_3\,, \\ V_2 &:= \mathsf{A}\,(\zeta_1 z_1 + \lambda\gamma_{01} z_1 + \gamma_{01}\zeta_1) + \mathsf{B}\,(\zeta_2 z_2 + \lambda\gamma_{02} z_2 + \gamma_{02}\zeta_2) \\ &\qquad\qquad + \mathsf{C}\,(\zeta_3 z_3 + \lambda\gamma_{03} z_3 + \gamma_{03}\zeta_3) \\ V_3 &:= z_1^2 + z_2^2 + z_3^2 + 2\boldsymbol{\gamma}_0 \cdot \boldsymbol{z}\,. \end{aligned} \tag{3.32}$$

Upright Spinning Top. Let

$$\mathsf{s}_0 = (r_0\,\boldsymbol{e}_3, -\boldsymbol{e}_3)\,,\ r_0 \neq 0\,.$$

This solution is representative of a permanent rotation of $\mathscr{B}$ around the principal axis $\boldsymbol{e}_3$ aligned with the vertical direction, and with the center of mass C at its highest position (*spinning top in its upright position*).

We follow the arguments of [1, §10], [18] and [11]. To this end, we begin to notice that, in our case, (3.32) furnishes:

$$\begin{aligned} V_1 &:= \mathsf{A}\,\zeta_1^2 + \mathsf{B}\,\zeta_2^2 + \mathsf{C}\,(\zeta_3^2 + 2r_0\zeta_3) - 2\beta^2\, z_3 = \text{const.}\,, \\ V_2 &:= \mathsf{A}\,\zeta_1 z_1 + \mathsf{B}\,\zeta_2 z_2 + \mathsf{C}\,(\zeta_3 z_3 + r_0 z_3 - \zeta_3) = \text{const.}\,, \\ V_3 &:= z_1^2 + z_2^2 + z_3^2 - 2z_3 = \text{const.}\,. \end{aligned} \tag{3.33}$$

<u>Case 1</u>: $\mathsf{A} = \mathsf{B}$ (*Lagrange Top*). Recalling that $C \in (O, \boldsymbol{e}_3)$, from $(3.18)_2$ we deduce the further first integral $\omega_3(t) = r_0 + \zeta_3(t) = \text{const}$. This implies that the function

$$V_4 := \zeta_3 \tag{3.34}$$

is constant along solutions to $(3.18)_2$–(3.19). The leading idea is now to take a suitable combination of the functions V_i, $i = 1, \dots, 4$, in a way that all linear terms cancel out, so that the resulting function, V, reduces to a quadratic form in the relevant variables. Since $\dot{V} = 0$ along solutions to $(3.18)_2$–(3.19), by Proposition 3.0.1, the stability of s_0 will be ensured by those conditions that make V positive definite. With this in mind, we choose

$$V := V_1 + 2\chi\, V_2 + (\mathsf{C}\, r_0\chi - \beta^2)V_3 + \mu\, V_4^2 + 2\mathsf{C}\,(\chi - r_0)V_4\,,$$

with χ and μ suitable real parameters that will be specified later on. Using, in the latter, (3.33) with $\mathsf{A} = \mathsf{B}$, and (3.34) we thus obtain

$$
\begin{aligned}
&V = Q_1 + Q_2 + Q_3 \\
&Q_1 := \mathsf{A}\,\zeta_1^2 + (\mathsf{C}\,r_0\,\chi - \beta^2)z_1^2 + 2\chi\,\mathsf{A}\,\zeta_1 z_1 \\
&Q_2 := \mathsf{A}\,\zeta_2^2 + (\mathsf{C}\,r_0\,\chi - \beta^2)z_2^2 + 2\chi\,\mathsf{A}\,\zeta_2 z_2 \\
&Q_3 := (\mathsf{C} + \mu)\,\zeta_3^2 + (\mathsf{C}\,r_0\,\chi - \beta^2)z_3^2 + 2\chi\,\mathsf{C}\,\zeta_3 z_3\,.
\end{aligned}
$$

Applying Sylvester's criterion, we deduce that both Q_1 and Q_2 are positive-definite if and only if there is $\chi \in \mathbb{R}$ such that

$$
\mathsf{A}\,\chi^2 - \mathsf{C}\,r_0\,\chi + \beta^2 < 0, \tag{3.35}
$$

which is indeed the case, provided the discriminant is positive, namely,

$$
r_0^2 > \frac{4\,\mathsf{A}\beta^2}{\mathsf{C}^2}\,. \tag{3.36}
$$

As for Q_3, again by the above criterion, we get that it is positive definite provided the following two conditions are met

$$
\mathsf{C} + \mu > 0\,, \quad (\mathsf{C} + \mu)(\mathsf{C}r_0\chi - \beta^2) - \chi^2\mathsf{C}^2 < 0\,.
$$

However, this last displayed inequality coincides with (3.35) with the choice

$$
\mu = \frac{\mathsf{C}}{\mathsf{A}}(\mathsf{C} - \mathsf{A})\,, \tag{3.37}
$$

in which case we also have $\mathsf{C} + \mu = \mathsf{C}^2/\mathsf{A} > 0$. We may therefore conclude that condition (3.36) ensures that V is positive definite, which, in turn, secures the stability of the steady-state motion s_0. As a matter of fact, the strict negation of (3.36), namely,

$$
r_0^2 < \frac{4\,\mathsf{A}\beta^2}{\mathsf{C}^2}\,. \tag{3.38}
$$

implies the instability of s_0. The proof of this property is established by a spectral stability analysis (Lyapunov's *"linearization principle"*) and goes as follows [11, Theorem 15.10.1]. Replacing the "perturbed motion" (3.31) with $\boldsymbol{\gamma}_0 \equiv -\boldsymbol{e}_3$, $\lambda \equiv -r_0$, back into $(3.18)_2$–(3.19), and recalling that $C \in (O, \boldsymbol{e}_3)$ and $\mathsf{A} = \mathsf{B}$, we show that the "perturbation" $(\boldsymbol{\zeta}, \boldsymbol{z})$ satisfies the following autonomous system

$$
\dot{\boldsymbol{X}} = \mathbb{A}\cdot\boldsymbol{X} + \boldsymbol{N}(\boldsymbol{X})\,,
$$

where $\boldsymbol{X} = (\zeta_1, \zeta_2, z_1, z_2)$, $\boldsymbol{N}$ is smooth with $|\boldsymbol{N}(\boldsymbol{X})|/|\boldsymbol{X}| \to 0$ as $|\boldsymbol{X}| \to 0$, and

$$
\mathbb{A} := \begin{bmatrix} 0 & \sigma & 0 & -\beta^2/\mathsf{A} \\ -\sigma & 0 & \beta^2/\mathsf{A} & 0 \\ 0 & 1 & 0 & r_0 \\ -1 & 0 & -r_0 & 0 \end{bmatrix}
$$

with $\sigma := (\mathsf{A} - \mathsf{C})/\mathsf{A}$. By a direct computation, one can then prove that if (3.38) holds, the four eigenvalues of $\mathbb{A}$ are of the type $\lambda_0, \overline{\lambda_0}, -\lambda_0, -\overline{\lambda_0}$, for some $\lambda_0 \in \mathbb{C}$ with both $\Re\{\lambda_0\}, \Im\{\lambda_0\} \neq 0$; see [11, p. 511] for details. As a consequence, two eigenvalues must have *positive* real part, which in turn, by a classical result due to LYAPUNOV [5, Corollary 6.1 in Chapter III], ensures that the state s_0 is unstable.

The above results can be then summarized in the following.

Theorem 3.2.2. *Let* $\mathsf{s}_0 := (r_0\, \boldsymbol{e}_3, -\boldsymbol{e}_3)$, $r_0 \neq 0$, *be a solution to* (3.26) *and assume* $\mathsf{A} = \mathsf{B}$ *(symmetric top, spinning in the upright position). Then if* (3.36) *holds,* s_0 *is stable, whereas if* (3.38) *holds,* s_0 *is unstable.*

Case 2: $\mathsf{C} > \mathsf{A}, \mathsf{B}$ (*Generic Top*). We follow the argument of [17]. As in the previous case, we will form a combination, V, of the functions V_i, $i = 1, 2, 3$, in (3.33) with the property that V is quadratic in a neighborhood of the origin, $\mathcal{U}$. Since $\dot{V} = 0$ along solutions to $(3.18)_1$–(3.19), by Proposition 3.0.1, the conditions ensuring that V is positive definite in $\mathcal{U}$ will also guarantee the stability of s_0. We thus choose

$$V := V_1 + 2\, r_0\, V_2 + (\mathsf{C}\, r_0^2 - \beta^2) V_3 + \eta\, V_3^2\,, \tag{3.39}$$

with η a parameter that will be fixed later on. Replacing the expressions of the V_i's from (3.33) back in (3.39) furnishes

$$\begin{aligned}
&V = \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3 + \mathcal{O}\\
&\mathcal{Q}_1 := \mathsf{A}\, \zeta_1^2 + 2\mathsf{A}\, r_0 \zeta_1 z_1 + (\mathsf{C}\, r_0^2 - \beta^2)\, z_1^2\\
&\mathcal{Q}_2 := \mathsf{B}\, \zeta_2^2 + 2\mathsf{B}\, r_0 \zeta_2 z_2 + (\mathsf{C}\, r_0^2 - \beta^2)\, z_2^2\\
&\mathcal{Q}_3 := \mathsf{C}\, \zeta_3^2 + 2\mathsf{C}\, r_0 \zeta_3 z_3 + (\mathsf{C}\, r_0^2 - \beta^2 + 4\,\eta)\, z_3^2\\
&\mathcal{O} := \eta\, (\boldsymbol{z}^4 - 4 z_3 \boldsymbol{z}^2)\,.
\end{aligned} \tag{3.40}$$

By Sylvester's criterion we prove at once that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both positive definite if

$$r_0^2 > \frac{\beta^2}{(\mathsf{C} - \mathsf{M})}\,, \quad \mathsf{M} := \max\{\mathsf{A}, \mathsf{B}\}\,. \tag{3.41}$$

As for $\mathcal{Q}_3$, again by that criterion, we show that it is positive definite for any admissible value of C, r_0 and β, if we pick, for example, $\eta = \beta^2/2$. Thus, choosing $\mathcal{U}$ so that $|\mathcal{O}| < \varepsilon\, \boldsymbol{z}^2$ for sufficiently small $\varepsilon > 0$, we conclude that if (3.41) holds, then V is positive definite in $\mathcal{U}$. We thus have proved the following.

Theorem 3.2.3. *Let* s_0 *be as in* Theorem 3.2.2 *and assume* $\mathsf{C} > \mathsf{A}, \mathsf{B}$ *(generic top, spinning in the upright position).* [3] *Then, if* (3.41) *holds,* s_0 *is stable.*

[3] Notice that, under the stated assumption on C, condition (3.39) is relevant when $\mathsf{A} \neq \mathsf{B}$ (asymmetric top). This because, otherwise, the stability requirement (3.36) is less restrictive and, therefore, more convenient.

Hanging Spinning Top. Another noteworthy steady-state solution "complementary" to that considered in the previous case is the one where $\mathscr{B}$ is imparted a uniform rotation around $\boldsymbol{e}_3$ with $\boldsymbol{e}_3$ still parallel to a_O but with its center of mass C in its lowest position (*hanging spinning top*). This motion is represented by the solution of the form [4]

$$\mathsf{s}_0 := (r_0\,\boldsymbol{e}_3, \boldsymbol{e}_3)\,, \quad r_0 \neq 0.$$

The functions in (3.32) specialize to the following ones:

$$\begin{aligned}
V_1 &:= \mathsf{A}\,\zeta_1^2 + \mathsf{B}\,\zeta_2^2 + \mathsf{C}\,(\zeta_3^2 + 2r_0\zeta_3) - 2\beta^2\,z_3\,,\\
\widehat{V}_2 &:= \mathsf{A}\,\zeta_1 z_1 + \mathsf{B}\,\zeta_2 z_2 + \mathsf{C}\,(\zeta_3 z_3 + r_0 z_3 + \zeta_3)\\
\widehat{V}_3 &:= z_1^2 + z_2^2 + z_3^2 + 2z_3
\end{aligned}$$

<u>Case 1</u>: $\mathsf{A} = \mathsf{B}$ (*Lagrange Top*). We recall that, in this situation, the further first integral (3.34) holds. Let

$$V := V_1 + 2\chi\,\widehat{V}_2 + (\beta^2 - \mathsf{C}\,r_0\chi)\widehat{V}_3 + \mu\,V_4^2 - 2\mathsf{C}\,(\chi + r_0)V_4\,.$$

Replacing in the latter the expressions of the V_i's and $\widehat{V}_i$'s, we obtain

$$\begin{aligned}
V &= \widehat{Q}_1 + \widehat{Q}_2 + \widehat{Q}_3\\
\widehat{Q}_1 &:= \mathsf{A}\,\zeta_1^2 + (\beta^2 - \mathsf{C}\,r_0\,\chi)z_1^2 + 2\chi\,\mathsf{A}\,\zeta_1 z_1\\
\widehat{Q}_2 &:= \mathsf{A}\,\zeta_2^2 + (\beta^2 - \mathsf{C}\,r_0\,\chi)z_2^2 + 2\chi\,\mathsf{A}\,\zeta_2 z_2\\
\widehat{Q}_3 &:= (\mathsf{C} + \mu)\,\zeta_3^2 + (\beta^2 - \mathsf{C}\,r_0\,\chi)z_3^2 + 2\chi\,\mathsf{C}\,\zeta_3 z_3\,.
\end{aligned}$$

By Sylvester's criterion, $\widehat{Q}_i$, $i = 1, 2$, are positive definite if there exists $\chi \in \mathbb{R}$ such that

$$\mathsf{A}\chi^2 + \mathsf{C}\,r_0\chi - \beta^2 < 0\,.$$

Since the discriminant, $\mathsf{C}^2 r_0^2 + 4\mathsf{A}\,\beta^2$, is always positive, this inequality can be satisfied for any given value of the physical parameters by choosing χ suitably. As for $\widehat{Q}_3$, it is positive definite if

$$\mathsf{C} + \mu > 0\,, \quad (\mathsf{C} + \mu)(\beta_2 - \mathsf{C}\,r_0\,\chi) - \chi^2\mathsf{C}^2 > 0\,,$$

which, by exactly the same argument used earlier on, are always satisfied with the choise of μ as in (3.37). We thus have proved the following.

Theorem 3.2.4. *Let* $\mathsf{s}_0 := (r_0\,\boldsymbol{e}_3, \boldsymbol{e}_3)$, $r_0 \neq 0$, *be a solution to* (3.26) *with* $\mathsf{A} = \mathsf{B}$ *(hanging spinning symmetric top). Then* s_0 *is always stable.*

[4] We assume the top is spinning, that's why we require $r_0 \neq 0$. However, the results stated in the next two theorems continue to hold (trivially) also in the case $r_0 = 0$.

Case 2: $\mathsf{A} \neq \mathsf{B}$. The stability of s_0 is now appropriately investigated by using the following combination of the above quantities:

$$V = V_1 - 2r_0\widehat{V}_2 + (\beta^2 + Cr_0^2)\widehat{V}_3 := \widehat{\mathcal{Q}}_1 + \widehat{\mathcal{Q}}_2 + \widehat{\mathcal{Q}}_3$$

where

$$\begin{aligned}
\widehat{\mathcal{Q}}_1 &:= \mathsf{A}\,\zeta_1^2 - 2r_0\mathsf{A}\,\zeta_1 z_1 + (\beta^2 + Cr_0^2)z_1^2\\
\widehat{\mathcal{Q}}_2 &:= \mathsf{B}\,\zeta_2^2 - 2r_0\mathsf{B}\,\zeta_2 z_2 + (\beta^2 + Cr_0^2)z_2^2\\
\widehat{\mathcal{Q}}_3 &:= \mathsf{C}\,\zeta_1^2 - 2r_0\mathsf{C}\,\zeta_3 z_3 + (\beta^2 + Cr_0^2)z_3^2\,.
\end{aligned}$$

Applying Sylvester's criterion we deduce that $\widehat{\mathcal{Q}}_3$ is positive definite, while $\widehat{\mathcal{Q}}_1$, $\widehat{\mathcal{Q}}_2$ enjoy the same property if and only if the following conditions are satisfied

$$r_0^2(\mathsf{C} - \mathsf{A}) + \beta^2 > 0\,,\quad r_0^2(\mathsf{C} - \mathsf{B}) + \beta^2 > 0\,.$$

We thus have the following theorem.

Theorem 3.2.5. *Let* $\mathsf{s}_0 := (r_0\,\boldsymbol{e}_3, \boldsymbol{e}_3)$, $r_0 \neq 0$, *be a solution to* (3.26) *with* $\mathsf{A} \neq \mathsf{B}$ *(hanging spinning asymmetric top). Then the following properties hold.*

(i) *If* $\mathsf{C} \geq \mathsf{A}, \mathsf{B}$, *then* s_1 *is always stable.*

(ii) *If either* $\mathsf{C} < \mathsf{A}, \mathsf{B}$, *or* $\mathsf{A} \leq \mathsf{C} < \mathsf{B}$, *or else* $\mathsf{B} \leq \mathsf{C} < \mathsf{A}$, *then* s_1 *is stable provided*

$$r_0^2 < \frac{\beta^2}{\mathsf{M} - \mathsf{C}}\,,$$

where $\mathsf{M} := \max\{\mathsf{B}, \mathsf{C}\}$.

Steady Precession. Our final objective is to present stability properties of motions in the SP classes defined in (3.29)–(3.30). A detailed study of these properties in the general case is beyond our scopes, and we refer the interested reader to [10, pp. 87–89]. Also for the application that we will develop later on in the case when the cavity is liquid-filled, we shall limit ourselves to analyze the stability of steady precessions in the class SP_1 (analogous considerations will hold for SP_2). We thus choose

$$\mathsf{s}_0 := (\lambda\,\boldsymbol{\gamma}_0, \boldsymbol{\gamma}_0 = \gamma_{02}\boldsymbol{e}_2 + \gamma_{03}\boldsymbol{e}_3)\,,\quad \gamma_{03} = \frac{\beta^2}{\lambda^2(\mathsf{B} - \mathsf{C})}\,,\quad |\gamma_{03}| < 1\,. \tag{3.42}$$

In such a case, the functions V_i's in (3.32) become

$$\begin{aligned}
V_1 &:= \mathsf{A}\,\zeta_1^2 + \mathsf{B}\,(\zeta_2^2 + +2\lambda\,\gamma_{02}\zeta_2) + \mathsf{C}\,(\zeta_3^2 + 2\lambda\gamma_{03}\zeta_3) - 2\beta^2\,z_3\,,\\
V_2 &:= \mathsf{A}\,\zeta_1 z_1 + \mathsf{B}\,(\zeta_2 z_2 + \lambda\gamma_{02}z_2 + \gamma_{02}\zeta_2) + \mathsf{C}\,(\zeta_3 z_3 + \lambda\gamma_{03}z_3 + \gamma_{03}\zeta_3)\\
V_3 &:= z_1^2 + z_2^2 + z_3^2 + 2\gamma_{01}z_1 + 2\gamma_{02}z_2\,.
\end{aligned} \tag{3.43}$$

Following [2], we consider the following function

$$V := V_1 - 2\,\lambda\, V_2 + \lambda^2\, \mathsf{B}\, V_3 + \mu\, V_3^2\,, \quad \mu > 0\,.$$

If we replace (3.43) in the latter and take into account the expression for γ_{03} in (3.42), we get that the contribution of the linear terms drops out and, more precisely, we find

$$\begin{aligned}
&V = \mathsf{Q}_1 + \mathsf{Q}_2 + \mathsf{Q}_3 + \mathsf{f}\\
&\mathsf{Q}_1 := \mathsf{A}\,\zeta_1^2 - 2\lambda\,\mathsf{A}\,\zeta_1 z_1 + \lambda^2 \mathsf{B}\, z_1^2\\
&\mathsf{Q}_2 := \mathsf{B}\,\zeta_2^2 - 2\lambda\,\mathsf{B}\,\zeta_2 z_2 + (\lambda^2 \mathsf{B} + \mu\,\gamma_{02}^2)\, z_2^2\\
&\mathsf{Q}_2 := \mathsf{C}\,\zeta_2^2 - 2\lambda\,\mathsf{C}\,\zeta_3 z_3 + (\lambda^2 \mathsf{B} + \mu\,\gamma_{03}^2)\, z_3^2\\
&|\mathsf{f}| < \varepsilon\,|\boldsymbol{z}|^2\,,
\end{aligned}$$

arbitrary small $\varepsilon > 0$. By applying Sylvester's criterion, we deduce at once that the Q_i's are positive definite if $\mathsf{B} > \mathsf{A}, \mathsf{C}$. Thus, restricting V to a sufficiently small neighborhood of $(\boldsymbol{\zeta} = \mathbf{0}, \boldsymbol{z} = \mathbf{0})$, with the help of Proposition 3.0.1 we deduce the following.

Theorem 3.2.6. *Steady precessions of the type* (3.42)*, namely, in the class* SP_1*, are stable, provided* $\mathsf{B} > \mathsf{A}, \mathsf{C}$.

In an entirely analogous way, one can show the following.

Theorem 3.2.7. *Steady precessions in the class* SP_2 *are stable, provided* $\mathsf{A} > \mathsf{B}, \mathsf{C}$.

Remark 3.2.8. From the results just proved it turns out that steady precessions satisfying the stated stability conditions must have $\gamma_{03} > 0$. This means that the $\boldsymbol{e}_3$-axis rotates around a_O with the center of mass C of $\mathcal{B}$ *below* the horizontal plane, P_O, passing through O ("*hanging steady precession*"). However, a stability analysis is also available for steady precessions with $\gamma_{03} < 0$, namely, when C is *above* P_O ("*upright steady precessions*"). For example, one can show [10, p. 88] that when $\mathsf{C} > B$, and $\mathsf{A} > \mathsf{B}$, the generic motion (3.42) is *unstable* if $\lambda^2 < \sqrt{3}\,\beta^2/\sqrt{\mathsf{B}(\mathsf{C}-\mathsf{B})}$.

3.2.1.2 Unsteady Motions.

The full class of these motions is obtained by solving $(3.18)_2$–(3.19) under prescribed initial conditions on the angular velocity, $\boldsymbol{\omega}$, and the "orientation" of the body, $\boldsymbol{\gamma}$. This problem, in its generality, has attracted the attention of many prominent mathematicians of the second half of the 19th and early 20th centuries, including S.V. Kovalevskaya, A.M. Lyapunov and T. Levi-Civita, who provided fundamental contributions to its resolution. We will not discuss any detail here, referring the interested reader to the comprehensive monograph [10].

What instead we would like to emphasize in this context is that, generically speaking, unsteady motions of a hollow heavy body $\mathscr{B}$ with a fixed point may present *permanent* complicated, and even "chaotic", characteristics that, as we shall see later on, eventually must disappear if the interior of the body is entirely filled with a viscous liquid.

To give a significant idea of how "complex" the motion can be, it is enough to observe, for example, that the center of mass C of $\mathscr{B}$ moves like the point mass of a *spherical pendulum*. Thus, as is well known, the motion of C, in general, is not necessarily periodic, with C describing a trajectory that lies in the zone between two horizontal concentric circles centered at points of the vertical axis passing through O; see, e.g., [14, Section 5.3].

Another, not less interesting example is furnished by the symmetric top when spinning in its upright position. As we have seen in the previous section, this motion is stable if and only if the angular velocity is sufficiently large (see (3.36)). This means, in particular, that if slightly perturbed, the top will still move with its axis, $\boldsymbol{e}_3$ in a neighborhood of the vertical axis passing through the fixed point O. However, it is very well known that the subsequent motion of the axis $\boldsymbol{e}_3$ (and of the top as a result) can be very complicated, with the generic of its points describing an intricate path, depending on the given initial conditions see [16, §204].

As we shall show in a later chapter, if the body contains a cavity entirely filled with a viscous liquid, involved motions of the above type can at most occur only during a *finite* interval of time, whose length may depend (all other parameters being fixed) on the viscosity of the liquid (the larger the viscosity the shorter the interval). After that, the liquid, thanks to its *incompressibility property*, will perform a strong stabilizing effect that, eventually, will bring the *coupled* system to a steady terminal state that, at times, may be even the rest.

3.2.2 Motion Around a Fixed Axis

The interesting situation occurs when the axis of rotation r is horizontal and $C \notin \mathsf{r}$ (*compound* or *physical pendulum*). Assuming, as usual, that the constraint is frictionless, we thus have that all reaction forces must be orthogonal to r. Consequently, for arbitrary $O \in \mathsf{r}$ and $\boldsymbol{e}_3$ parallel to r, it follows $\boldsymbol{\tau}_O \cdot \boldsymbol{e}_3 = 0$. Furthermore, clearly, $\boldsymbol{\omega}(t) = \omega(t)\,\boldsymbol{e}_3$, so that $\boldsymbol{e}_3 \cdot \mathbb{J}_O \cdot \boldsymbol{\omega} = \mathsf{C}\,\omega$, where C is the moment of inertia of $\mathscr{B}$ with respect to r. Finally, choosing O and $\boldsymbol{e}_1$ in such a way that $\boldsymbol{x}_C = \ell\,\boldsymbol{e}_1$, $\ell > 0$, we deduce that the torque due to the gravity is given by $\boldsymbol{M}_O = M_{\mathcal{B}}\,g\,\ell\,\boldsymbol{e}_1 \times \boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, 0)$ satisfies (3.19). Therefore, projecting $(3.1)_2$ along $\boldsymbol{e}_3$ we obtain that the motion of $\mathscr{B}$ is governed by the following set of equations

$$\dot{\omega} = \alpha^2\,\gamma_2\,, \quad \dot{\gamma}_1 = \gamma_2\,\omega\,, \quad \dot{\gamma}_2 = -\gamma_1\,\omega\,, \quad \gamma_1^2 + \gamma_2^2 = 1\,, \tag{3.44}$$

with $\alpha^2 := M_{\mathcal{B}}\,g\,\ell/\mathsf{C}$.

From (3.44) we immediately infer that two and only two steady-state motions ($\dot{\omega} = 0$) are allowed. They correspond to the *equilibrium* configurations where

$\omega \equiv 0$ and either $\gamma_1 = 1$ (C at its lowest position) or $\gamma_1 = -1$ (C at its highest position). The stability of these equilibria is very simply studied with the help of Proposition 3.0.1 and Proposition 3.0.2. Indeed, from (3.44) we deduce the following first (total energy) integral

$$T - U := \tfrac{1}{2}\,\omega^2 - \alpha^2\,\gamma_1 = \text{const}\,. \tag{3.45}$$

Therefore, denoting by $(0, \boldsymbol{\gamma}_0 = \gamma_0 \boldsymbol{e}_1)$ the generic equilibrium and by $(\boldsymbol{\zeta}(t), \boldsymbol{z}(t) + \boldsymbol{\gamma}_0)$ a corresponding perturbed motion, from $(3.44)_4$ and (3.45) we deduce that the following functions

$$V_1 := \zeta^2 - 2\alpha^2 z_1\,, \quad V_2 := z_1^2 + z_2^2 + 2\gamma_0 z_1\,,$$

must be constant as well. Now, if $\gamma_0 = 1$ (C in its lowest position) we choose

$$V := V_1 + \alpha^2\,V_2 = \zeta^2 + \alpha^2(z_1^2 + z_2^2)$$

from which, in view of Proposition 3.0.1, we immediately deduce the stability of such equilibrium configuration. On the other hand, we can also readily prove that the other equilibrium $(0, \boldsymbol{\gamma}_0 = -\boldsymbol{e}_1)$ is unstable. To this end, choosing

$$V := \zeta\, z_2\,,$$

from (3.44) we deduce

$$\dot V = \alpha^2\, z_2^2 + \zeta^2 - z_1\,\zeta^2.$$

Thus, if we define

$$U := \{(\zeta, \boldsymbol{z}) \in \mathbb{R} \times \mathbb{R}^2 :\ |z_1| < \tfrac{1}{2}\}\,, \quad U_1 := \{(\zeta, \boldsymbol{z}) \in \mathbb{R} \times \mathbb{R}^3 :\ \zeta\, z_2 > 0\}\,,$$

we recognize at once that all assumptions of Proposition 3.0.2 are recovered thus showing that $(0, \boldsymbol{\gamma}_0 = -\boldsymbol{e}_1)$ is unstable.

We end this section with a few –but important for future reference– comments about the *generic* motion of the compound pendulum. As is well known, its semiquantitative behavior can be obtained from (3.44) by means of the so-called “Weierstrass argument,” based on the study of a suitable elliptic integral [14, §5.2]. In particular, introducing the non-dimensional quantity $\eta := (T(0) - U(0))/\alpha^2$ it can be shown that the motion of the compound pendulum will fall in one of the following categories: (i) Equilibrium in the above-mentioned configurations when $\eta = \pm 1$; (ii) Oscillation between two symmetric configurations if $\eta \in (-1, 1)$, and (iii) Continuous rotation around r if $\eta > 1$.

Chapter 4

Cavity Filled with an Inviscid Liquid

In this chapter we begin to study the motion of the rigid body, $\mathscr{B}$, when its interior cavity, $\mathscr{C}$, is entirely filled with a liquid. Following the pioneering approach of Stokes [19, §13] and Zhukowskiy [22] we will first consider the situation when the liquid is inviscid ($\mu = 0$) and its motion is irrotational, namely, the vorticity field is identically zero. In the case when $\mathscr{C}$ is simply connected –an assumption we shall keep throughout, for simplicity– the latter implies that the (absolute) velocity field of the liquid is potential-like. Under this hypothesis, one shows that the dynamics of the coupled system body-liquid, $\mathscr{S}$, is reduced to that of a *single* "transformed" rigid body, whose inertia tensor is the sum of the inertia tensor $\mathbb{J}_O$ of $\mathscr{B}$ and a symmetric, *non-negative* definite tensor $\mathbb{J}_O^*$ that depends only on the density of the liquid and the shape of the cavity. In other words, the dynamics of the *coupled* system liquid-body becomes that of a system with *finite* degrees of freedom. One important consequence of this result is that the stability analysis of the steady-state motions of $\mathscr{S}$ is entirely analogous to that performed in the previous chapter for the rigid body with an empty cavity, by just replacing $\mathbb{J}_O$ with $\mathbb{J}_O + \mathbb{J}_O^*$.

Another important aspect is that, already for this simplified model, we are able to show, in specific instances, that the presence of the liquid can dramatically change the dynamics of the body. As a matter of fact, we will prove that in fairly common cases (simple shapes of body and cavity) the liquid can totally *alter the stability properties* of $\mathscr{B}$, *by rendering stable certain steady-state motions* (permanent rotations) *that are unstable in absence of the liquid, and vice-versa.*

It should now be observed that, while the *instability* conditions formulated in the class of irrotational liquid flow, $\mathfrak{I}$, continue clearly to be valid in a more general class of motions, the *stability* results derived in the same class may not be completely satisfactory. Stated differently, a steady-state motion that is stable in the class $\mathfrak{I}$ may turn out to be unstable in a larger class of perturbations. These considerations then lead us to investigate the stability properties in a class of liquid motions, $\mathfrak{C}$, that is physically relevant and, in principle, strictly contains $\mathfrak{I}$.

An entirely natural choice is to pick $\mathfrak{C}$ to be the class of solutions to (2.20)–(2.24) that satisfy (in a suitable sense) conservation of total energy and (axial) angular momentum. We shall thus perform a stability analysis of steady state motions in $\mathfrak{C}$ by using a simple variant of Proposition 3.0.1 (see Proposition 4.5.5) where now the stability of the liquid component is measured in terms of the magnitude of its kinetic energy. In this manner we are able to show that most *stability* results proved in Section 3.2.1.2 for the heavy body with a fixed point O, continue to hold in the class $\mathfrak{C}$, provided we replace $\mathbb{J}_O$ with the inertia tensor $\mathbb{I}_O$ of the coupled system $\mathscr{S}$. As far as steady-state inertial motions, we can show that permanent rotations occurring around the central axis of *maximum* moment of inertia are stable but, unlike Theorem 3.1.1, we cannot draw any conclusion about those occurring around the central axis of *minimum* moment of inertia. As a matter of fact, as we detailed in the introductory chapter, the seminal experimental work of Lord Kelvin strongly suggests that permanent rotations occurring around the axis of minimum moment of inertia are unstable, no matter how large the magnitude of the initial angular velocity. Seemingly, the inviscid theory is not able to furnish an explanation of this phenomenon. However, as we will show in a later chapter, the same will find a full and rigorous mathematical interpretation if we assume that the cavity is entirely filled with a *viscous* Navier-Stokes liquid.

4.1 Irrotational Flow

We will suppose throughout that the the volume $\mathcal{C}$ occupied by the cavity is entirely filled with an inviscid ($\mu = 0$) liquid $\mathscr{L}$. We shall next admit that the generic motion of $\mathscr{L}$ is irrotational, namely, the vorticity field associated with the absolute velocity field $\boldsymbol{u}$ is identically vanishing:

$$\operatorname{curl} \boldsymbol{u}(x,t) = \boldsymbol{0}\,, \quad (x,t) \in \mathcal{C} \times (0,\infty)\,. \tag{4.1}$$

Under the further assumption that *the cavity is simply connected* [1] condition (4.1) implies the existence of a single-valued function $\varphi = \varphi(x,t)$ such that

$$\boldsymbol{u} = \nabla\varphi\,. \tag{4.2}$$

From (4.2) and $(2.17)_{2,3}$ we immediately deduce that the function φ must satisfy the following Neumann problem at each time $t > 0$

$$\Delta\varphi = 0 \text{ in } \mathcal{C}\,, \quad \frac{\partial\varphi}{\partial n} = (\boldsymbol{\xi}_O + \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \boldsymbol{n} \;\text{ at } \partial\mathcal{C}\,. \tag{4.3}$$

Moreover, observing that

$$(\nabla\varphi - \boldsymbol{\xi}_O - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\nabla\varphi = \tfrac{1}{2}\nabla(\nabla\varphi)^2 - \nabla[(\boldsymbol{\xi}_O - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla\varphi] - \boldsymbol{\omega} \times \boldsymbol{x} \cdot \nabla\varphi\,,$$

[1] This hypothesis is made for the sake of simplicity. Most of the basic properties that we shall prove will continue to hold also in the more general case with the appropriate modification.

we infer that $(2.17)_1$ (with $\mu = 0$) is satisfied by choosing

$$\widetilde{p} = -\frac{\partial \varphi}{\partial t} + (\boldsymbol{\xi}_O - \boldsymbol{\omega} \times \boldsymbol{x}) \cdot \nabla \varphi - \tfrac{1}{2}(\nabla \varphi)^2 + p_0 \,, \tag{4.4}$$

where $p_0 = p_0(t)$ is arbitrary. Summarizing, we may state that the motion of a rigid body with a simply-connected cavity entirely filled with an inviscid liquid is governed, under the assumption (4.1), by (4.3) along with (2.18)–(2.19) that now take the form

$$\begin{aligned} &M\,\dot{\boldsymbol{\xi}}_G + M\,\boldsymbol{\omega} \times \boldsymbol{\xi}_G = \boldsymbol{F} + \boldsymbol{\Phi} \\ &\dot{\boldsymbol{A}}_O + \boldsymbol{\omega} \times \boldsymbol{A}_O + M\,\boldsymbol{\xi}_O \times \boldsymbol{\xi}_G = \boldsymbol{M}_O + \boldsymbol{\tau}_O \,, \end{aligned} \tag{4.5}$$

with

$$\boldsymbol{A}_O := \mathbb{J}_O \cdot \boldsymbol{\omega} + M_{\mathcal{B}} \boldsymbol{x}_C \times \boldsymbol{\xi}_O + \int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \nabla \varphi \,. \tag{4.6}$$

The resolution of (4.3)–(4.6), under suitable assumption on the constraints, can be addressed as follows. From (4.3) we solve for φ as a functional of $\boldsymbol{\xi}_O$ and $\boldsymbol{\omega}$, then replace it back into (4.6) and solve (4.5) for $\boldsymbol{\xi}_O$ and $\boldsymbol{\omega}$. This is, basically, the procedure adopted by ZHUKOWSKIY, and that we shall present in the next section.

4.2 Zhukowskiy Potentials. Reduction to a System with Finite Degrees of Freedom

One of the important achievements of ZHUKOVSKIY's work is the proof that, under the irrotational assumption (4.1), the motion of the coupled system liquid-solid can be reduced to that of a suitable mechanical system having a finite number of degrees of freedom. To show this, we begin to notice that a solution to (4.3) can be sought in the form

$$\begin{aligned} &\varphi = \phi + \psi \,, \\ &\Delta \phi = 0 \text{ in } \mathcal{C} \,, \quad \frac{\partial \phi}{\partial n} = \boldsymbol{\xi}_O \cdot \boldsymbol{n} \;\text{ at } \partial\mathcal{C} \,, \\ &\Delta \psi = 0 \text{ in } \mathcal{C} \,, \quad \frac{\partial \psi}{\partial n} = \boldsymbol{\omega} \times \boldsymbol{x} \cdot \boldsymbol{n} \;\text{ at } \partial\mathcal{C} \,. \end{aligned} \tag{4.7}$$

Thus, introducing the six *time-independent* functions $\phi_i = \phi_i(x)$, $\psi_i = \psi_i(x)$, $i = 1, 2, 3$ with the properties

$$\begin{aligned} &\Delta \phi_i = 0 \text{ in } \mathcal{C} \,, \quad \frac{\partial \phi_i}{\partial n} = n_i \;\text{ at } \partial\mathcal{C} \,, \\ &\Delta \psi_i = 0 \text{ in } \mathcal{C} \,, \quad \frac{\partial \psi_i}{\partial n} = \boldsymbol{e}_i \times \boldsymbol{x} \cdot \boldsymbol{n} \;\text{ at } \partial\mathcal{C} \,, \end{aligned} \tag{4.8}$$

we recognize that the (unique) solutions to $(4.7)_{2,3}$ have the following expressions

$$\phi(x,t) = \xi_{Oi}(t)\,\phi_i(x)\,, \quad \psi(x,t) = \omega_i(t)\,\psi_i(x)\,. \tag{4.9}$$

The functions $\{\phi_i, \psi_i\}$ are independent of the liquid density and depend (at most) on the "shape" of the cavity. They are often referred to as *Zhukowskiy potentials*.

We next observe that the fields ϕ_i are easily determined. In fact, for any cavity $\mathcal{C}$, they are of the form

$$\phi_i(x) = x_i\,, \ i = 1, 2, 3\,,$$

from which, it easily follows that

$$\int_{\mathcal{C}} \rho\,\boldsymbol{x} \times \nabla\phi = \int_{\mathcal{C}} \rho\,\boldsymbol{x} \times \boldsymbol{\xi}_O\,. \tag{4.10}$$

On the other hand, the form of the fields ψ_i is, in general, more involved, and they can be obtained in closed form (or known within a good approximation) only for special "shapes" of the cavity (see [13, pp. 53–63] and also next section). In any case, they enjoy a number of fundamental properties that we would like to state in a very general form. To this end, we recall that, by well-known results, if $\mathcal{C}$ Lipschitz, then the Neumann problem in $(4.8)_{3,4}$ has a unique weak solution $\psi_i \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$, where the boundary condition is achieved in the sense of $W^{-1/2,2}(\partial\mathcal{C})$ []. We have the following.

Lemma 4.2.1. *Let $\mathcal{C}$ be Lipschitz and let ψ_i be the weak solution to $(4.8)_{3,4}$. Define the second-order tensor $\mathbb{J}^*_O$ through its components in the base $\{\boldsymbol{e}_i\}$ as follows* [2]

$$(\mathbb{J}^*_O)_{ij} = \int_{\partial\mathcal{C}} \rho\,\psi_i\,\frac{\partial\psi_j}{\partial n}\,, \quad i,j = 1, 2, 3. \tag{4.11}$$

Then, $\mathbb{J}^*_O$ is symmetric, and positive semidefinite:

$$\boldsymbol{a} \cdot \mathbb{J}^*_O \cdot \boldsymbol{a} \geq 0\,, \quad \textit{for all } \boldsymbol{a} \in \mathbb{R}^3 - \{\boldsymbol{0}\}\,. \tag{4.12}$$

In addition, $\mathbb{J}^*_O$ is positive definite if there exist points $x_i \in \partial\mathcal{C}$, $i = 1, 2, 3$ such that the vectors $(\boldsymbol{x}_1 \times \boldsymbol{n}(x_1), \boldsymbol{x}_2 \times \boldsymbol{n}(x_2), \boldsymbol{x}_3 \times \boldsymbol{n}(x_3))$ are linearly independent, whereas $\mathbb{J}^*_O$ is identically vanishing if $\mathcal{C}$ is a ball centered at O.

Proof. From $(4.8)_{3,4}$ it follows, in particular, that $\nabla\psi_i$ is identically vanishing if and only if $\boldsymbol{n} \times \boldsymbol{x} = \boldsymbol{0}$ for all $x \in \partial\mathcal{C}$, which proves the last assertion. We next show the symmetry of the tensor $\mathbb{J}^*_O$. Actually, multiplying both sides of $(4.8)_3$ by ψ_j and integrating by parts we get

$$(\mathbb{J}^*_O)_{ij} = \int_{\mathcal{C}} \nabla\psi_i \cdot \nabla\psi_j\,, \quad i,j = 1, 2, 3\,, \tag{4.13}$$

[2] The surface integral on the right-hand side of (4.11) should be interpreted, here and in the following, as the duality pair $W^{1/2,2}(\partial\mathcal{C}) \to W^{-1/2,2}(\partial\mathcal{C})$. We prefer to use this notation for sake of simplicity.

which delivers the desired property. Let $\boldsymbol{a} = \alpha_i \boldsymbol{e}_i$ and set

$$\overline{\psi} := \alpha_i \psi_i \,.$$

Multiplying both sides of $(4.8)_3$ by α_i and summing over the index i, we obtain $\Delta\overline{\psi} = 0$ in $\mathcal{C}$. Multiplying the latter equation by $\overline{\psi}$ and integrating over $\mathcal{C}$ we thus deduce

$$\|\nabla\overline{\psi}\|_2^2 = \sum_{i,j=1}^{3} \alpha_i \alpha_j \int_{\partial\mathcal{C}} \psi_i \frac{\partial \psi_j}{\partial n} \equiv \frac{1}{\rho}\, \boldsymbol{a} \cdot \mathbb{J}_O^* \cdot \boldsymbol{a}\,,$$

which proves (4.12). Moreover, if $\boldsymbol{a}\cdot\mathbb{J}_O^*\cdot\boldsymbol{a} = 0$, from the the last displayed equation we get $\overline{\psi} = \text{const.}$, which furnishes $\boldsymbol{a}\cdot\boldsymbol{x}\times\boldsymbol{n} = 0$, for all $x \in \partial\mathcal{C}$. This condition, under the stated assumptions, in turn implies $\boldsymbol{a} = \boldsymbol{0}$, and the proof of the lemma is completed.

□

We now analyze some consequences of Lemma 4.2.1. To this end, we notice that, by (4.9),

$$\int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \nabla\psi = \omega_i \int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \nabla\psi_i\,,$$

so that, integrating by parts,

$$\int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \nabla\psi = \omega_i \int_{\partial\mathcal{C}} \rho\, \psi_i \boldsymbol{x} \times \boldsymbol{n}\,. \tag{4.14}$$

Taking into account that

$$\boldsymbol{x} \times \boldsymbol{n} \cdot \boldsymbol{e}_j = \frac{\partial \psi_j}{\partial n} \quad \text{at } \partial\mathcal{C}$$

by (4.14) and Lemma 4.2.1 we deduce

$$\int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \nabla\psi = \mathbb{J}_O^* \cdot \boldsymbol{\omega}\,. \tag{4.15}$$

From (4.10), (4.15) and (2.24) we conclude the following important result basically due to ZHUKOWSKIY.

Theorem 4.2.2. *Suppose the cavity $\mathcal{C}$ is Lipschitz and simply connected and that the motion of the liquid is irrotational. Then the total angular momentum of the coupled system body-liquid with respect to the point O is given by*

$$\boldsymbol{A}_O = \mathbb{J}_O^1 \cdot \boldsymbol{\omega} + M\, \boldsymbol{x}_G \times \boldsymbol{\xi}_O\,, \tag{4.16}$$

where $\mathbb{J}_O^1 := \mathbb{J}_O + \mathbb{J}_O^*$, with $\mathbb{J}_O^*$ defined in (4.11).

Remark 4.2.3. Denote by $\mathscr{B}^*$, a rigid body having the same density as the liquid and inertia tensor $\mathbb{J}^*_O$. Such a body was called by Stokes *equivalent body* [19, p. 153]. Then, the fundamental consequence of Theorem 4.2.2 is that *provided we restrict ourselves to irrotational flow*, the liquid can be replaced by an equivalent body, without affecting the motion of the coupled system $\mathscr{S}$. More precisely, the study of the motion of $\mathscr{S}$ can be performed according to the following procedure. At first, for a given cavity, we solve the Neumann problem $(4.8)_{3,4}$. As soon as the functions ψ_i are obtained, the motion of $\mathscr{S}$ is reduced to that of the rigid body, $\mathscr{B}^1$, governed by (4.5) with the "modified" angular momentum (4.16), where the inertia tensor $\mathbb{J}_O$ of $\mathscr{B}$ is replaced by the tensor $\mathbb{J}^1_O := \mathbb{J}_O + \mathbb{J}^*_O$. The body $\mathscr{B}^1$, obtained by rigidly "connecting" $\mathscr{B}$ with the equvalent body $\mathscr{B}^*$, was called by Zhukovskiy *transformed body*, and we shall accordingly call $\mathbb{J}^1_O$ the "transformed inertia tensor." Therefore, the contribution of the liquid to the motion of the body relies all in the tensor quantity $\mathbb{J}^*_O$. For those cavities where $\mathbb{J}^*_O$ vanishes, the liquid has no influence on the motion of $\mathscr{B}$, like, for instance, when the cavity is a ball. However, as we shall show in the Section 4.4, there are significant cases where $\mathbb{J}^*_O$ is not zero and the dynamics of $\mathscr{B}$ is dramatically modified. Finally, we observe that, once the motion of the transformed body is resolved, namely, $\boldsymbol{\omega}$ and $\boldsymbol{\xi}$ are found, then we can resolve for the motion of the liquid as well. In fact, its (absolute) velocity field is provided by (4.2) with

$$\varphi(x,t) = \xi_{Oi}(t)x_i + \omega_i(t)\,\psi_i(x)\,,$$

and ψ_i solving $(4.8)_2$, while the corresponding pressure field is given in (4.4).

4.3 Ellipsoidal Cavity

In order to investigate how the presence of the liquid may effectively modify the motion of the rigid body, it is necessary to compute explicitly Zhukovskiy's potentials ψ_i. In general, this is, of course, a hopeless task. However, if the shape of the cavity is not too complicated, one is able to express these functions in a simple closed form [13, pp. 52–63]. Objective of this section is to evaluate the potentials ψ_i and the corresponding tensor $\mathbb{J}^*_O$ in the case when the cavity is an ellipsoid. In doing this, we shall closely follow the elegant method of Lamb [9, §110], successively employed by Zhukovskiy [22, §12].

Assume the cavity $\mathcal{C}$ is given by

$$\mathcal{C} := \left\{\boldsymbol{x} \in \mathbb{R}^3 :\ F(x_1,x_2,x_3) := \sum_{i=1}^{3} \frac{x_i^2}{\alpha_i^2} - 1 < 0\right\},$$

where $\alpha_i > 0$, $i = 1,2,3$. We then get that the unit normal $\boldsymbol{n}$ at $\partial\mathcal{C}$ has components

$$n_i = \frac{1}{|\nabla F|}\frac{\partial F}{\partial x_i} = \frac{2}{|\nabla F|}\frac{x_i}{\alpha_i^2}\,.$$

Consequently,

$$\boldsymbol{e}_1 \times \boldsymbol{x} \cdot \boldsymbol{n} = \frac{1}{|\nabla F|} \Big(\frac{1}{\alpha_3^2} - \frac{1}{\alpha_2^2} \Big) x_2 x_3 \,.$$

This suggests to choose ψ_1 as follows

$$\psi_1(x_1, x_2, x_3) = A_1 \, x_2 x_3 \,,$$

where A_1 is a constant to be determined later. Clearly, ψ_1 is harmonic in the whole of $\mathbb{R}^3$ and, in particular, in $\mathcal{C}$. Moreover, since

$$\frac{\partial \psi_1}{\partial n} \equiv \boldsymbol{n} \cdot \nabla \psi_1 = \frac{2\,A}{|\nabla F|} \Big(\frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} \Big) x_2 x_3$$

by imposing the boundary condition $(4.8)_4$ we find $A_1 = (\alpha_2^2 - \alpha_3^2)/(\alpha_2^2 + \alpha_3^2)$. Arguing in exactly the same way, we may find also ψ_2 and ψ_3, and thus conclude with the following expression for the potentials

$$\begin{aligned} &\psi_1 = A_1 \, x_2 x_3 \,, \quad \psi_2 = A_2 \, x_3 x_1 \,, \quad \psi_3 = A_3 \, x_2 x_1 \\ &A_1 = \frac{\alpha_2^2 - \alpha_3^2}{\alpha_2^2 + \alpha_3^2} \,, \quad A_2 = \frac{\alpha_3^2 - \alpha_1^2}{\alpha_1^2 + \alpha_3^2} \,, \quad A_3 = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2} \,.. \end{aligned} \tag{4.17}$$

Our next goal is to use (4.17) to determine the components of the tensor $\mathbb{J}_O^*$, with O center of the ellipsoid, in the base with origin at O and axes directed along the principal axes of the ellipsoid. To this end, we observe that from (4.13) and (4.17) it follows that

$$\begin{aligned} &(\mathbb{J}_O^*)_{11} = A_1^2 \int_{\mathcal{C}} \rho \, (x_2^2 + x_3^2) \,, \quad (\mathbb{J}_O^*)_{22} = A_2^2 \int_{\mathcal{C}} \rho \, (x_1^2 + x_3^2) \,, \quad (\mathbb{J}_O^*)_{33} = A_3^2 \int_{\mathcal{C}} \rho \, (x_1^2 + x_2^2) \,, \\ &(\mathbb{J}_O^*)_{ij} = A_i A_j \int_{\mathcal{C}} x_i x_j \,, \quad i \neq j \,. \end{aligned}$$

We also recall the following well-known formulas (e.g. [12, §48])

$$\begin{aligned} &\int_{\mathcal{C}} (x_1^2 + x_2^2) = \tfrac{4}{15} \, \pi \, \alpha_1 \alpha_2 \alpha_3 (\alpha_1^2 + \alpha_2^2) \,, \\ &\int_{\mathcal{C}} (x_2^2 + x_3^2) = \tfrac{4}{15} \, \pi \, \alpha_1 \alpha_2 \alpha_3 (\alpha_2^2 + \alpha_3^2) \,, \\ &\int_{\mathcal{C}} (x_1^2 + x_3^2) = \tfrac{4}{15} \, \pi \, \alpha_1 \alpha_2 \alpha_3 (\alpha_1^2 + \alpha_3^2) \,. \end{aligned} \tag{4.18}$$

Therefore, taking into account that, denoting by m the mass of the liquid, we have

$$m = \rho \, \frac{4}{3} \, \alpha_1 \alpha_2 \alpha_3 \,, \tag{4.19}$$

and that, in view of elementary symmetry properties, the off-diagonal components vanish, we conclude

$$\begin{aligned} \mathsf{A}^* &:= (\mathbb{J}_O^*)_{11} = \frac{m}{5}\,\frac{(\alpha_2^2-\alpha_3^2)^2}{\alpha_2^2+\alpha_3^2} \\ \mathsf{B}^* &:= (\mathbb{J}_O^*)_{22} = \frac{m}{5}\,\frac{(\alpha_3^2-\alpha_1^2)^2}{\alpha_3^2+\alpha_1^2} \\ \mathsf{C}^* &:= (\mathbb{J}_O^*)_{33} = \frac{m}{5}\,\frac{(\alpha_1^2-\alpha_2^2)^2}{\alpha_1^2+\alpha_2^2} \\ (\mathbb{J}_O^*)_{ij} &= 0\,, \quad i \neq j\,. \end{aligned} \tag{4.20}$$

Notice that, in particular, in the frame with the origin at the center of the ellipsoid and axes directed along the principal axes of the ellipsoid the tensor $\mathbb{J}_O^*$ turns out to be diagonal.

Remark 4.3.1. It is interesting to compare the principal moments A^*, B^* and C^* of the "equivalent body" $\mathscr{B}^*$ (see Remark 4.2.3) with those A_*, B_* and C_* of the liquid. From (4.18) and (4.19) we immediately deduce

$$\mathsf{A}_* = \frac{m}{5}\,(\alpha_2^2+\alpha_3^2)\,, \quad \mathsf{B}_* = \frac{m}{5}\,(\alpha_3^2+\alpha_1^2)\,, \quad \mathsf{C}_* = \frac{m}{5}\,(\alpha_1^2+\alpha_2^2)\,,$$

which implies

$$\mathsf{A}^* < \mathsf{A}_*\,, \quad \mathsf{B}^* < \mathsf{B}_*\,, \quad \mathsf{C}^* < \mathsf{C}_*\,.$$

As a result, the moment of inertia of the equivalent body with respect to *any* axis passing through the point O is strictly smaller than the analogous quantity evaluated for the liquid. In fact, as shown by ZHUKOWSKIY [22, §9], *this property continues to hold for cavities of arbitrary shape.*

4.4 On the Stability of Steady-State Motions

Another fundamental consequence of Theorem 4.2.2 (see also Remark 4.2.3), is that the stability of steady-state motions of the coupled system body-liquid when the cavity is entirely filled with an inviscid liquid, may be reduced to the same problem for the "transformed body." at least *if we confine ourselves to the case of irrotational flow*. Thus, taking the frame $\{O, \boldsymbol{e}_i\}$ as principal frame of inertia for the transformed inertia tensor $\mathbb{J}_O^1$, and denoting by A^1, B^1 and C^1 the corresponding principal moments of inertia, we may conclude with the following.

Theorem 4.4.1. *All stability/instability theorems established in* Section 3.1.1 *and* Section 3.2.1 *continue to hold when the cavity is entirely filled with an inviscid liquid, if we formally replace in their statements the quantities* A, B *and* C *with* A^1, B^1 *and* C^1, *respectively, and we restrict ourselves to irrotational flow of the liquid.*

Now, as we know, the hypotheses of most of the results mentioned in Theorem 4.4.1 are based on the relative magnitude of the principal moments of inertia. However, as somehow expected, the relative magnitude of the "transformed" A^1, B^1 and C^1 need not be the same as that of the analogous A, B and C evaluated when the cavity is empty. Consequently, the stability properties of the coupled system can be, in principle, entirely different than the analogous ones with an empty cavity. In order to show that this can be indeed the case, let us suppose that $\mathscr{B}$ is a solid ellipsoidal shell of constant density, $\widehat{\rho}$, contained between the two ellipsoids of equations

$$\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1 \quad \text{and} \quad \sum_{i=1}^{3} \frac{x_i^2}{\alpha_i^2} = 1\,, \tag{4.21}$$

respectively, with $\alpha_i := \eta\, a_i$, $\eta \in (0,1)$. We take O coinciding with the center of both ellipsoids and $\boldsymbol{e}_i$ in the direction of the semi-axis a_i. Due to the geometric symmetry property of $\mathscr{B}$ and its uniform distribution of mass, we have that $O \equiv C$ and $\{O, \boldsymbol{e}_i\}$ is a principal (central) frame of inertia for $\mathscr{B}$. The principal moments of inertia A, B and C are easily calculated. Actually, they are the differences between the analogous quantity computed for the larger (A_L, B_L and C_L) and smaller (A_S, B_S and C_S) ellipsoid, with the latter assumed to have the same density $\widehat{\rho}$. Thus, observing that

$$M_{\mathcal{B}} = \widehat{\rho}\,\frac{4}{3}\,\pi(a_1a_2a_3 - \alpha_1\alpha_2\alpha_3) = \widehat{\rho}\,\frac{4}{3}\,\pi\left(1-\eta^3\right) a_1a_2a_3\,,$$

with the help of (4.18) (with $\alpha_1 \equiv a_i$) and (3.2), we deduce

$$\begin{aligned} \mathsf{A} = \mathsf{A}_L - \mathsf{A}_S \; &= \frac{4}{15}\pi\,\widehat{\rho}\,a_1a_2a_3(a_2^2+a_3^2) - \frac{4}{15}\pi\,\widehat{\rho}\,\eta^5 a_1a_2a_3\,(a_2^2+a_3^2) \\ &= \frac{1}{5}\gamma\; M_{\mathcal{B}}\,(a_2^2+a_3^2)\,, \quad \gamma := \frac{1-\eta^5}{1-\eta^3}\,, \end{aligned} \tag{4.22}$$

and, likewise,

$$\begin{aligned} \mathsf{B} \; &= \mathsf{B}_L - \mathsf{B}_S = \frac{1}{5}\gamma\, M_{\mathcal{B}}\,(a_1^2+a_3^2) \\ \mathsf{C} \; &= \mathsf{C}_L - \mathsf{C}_S = \frac{1}{5}\gamma\, M_{\mathcal{B}}\,(a_1^2+a_2^2)\,. \end{aligned} \tag{4.23}$$

Assume, to fix the ideas,

$$a_3 < a_2 < a_1\,; \tag{4.24}$$

we then have

$$\mathsf{A} < \mathsf{B} < \mathsf{C}\,. \tag{4.25}$$

Therefore, the moment of inertia of the system around $(O, \boldsymbol{e}_3)$ is maximum and so, by Theorem 3.1.1, we may conclude that *in absence of external forces* (inertial motion), *the permanent rotation around* $(O, \boldsymbol{e}_3)$ *is stable, whereas that around* $(O, \boldsymbol{e}_2)$ *is unstable.*

We now suppose that the liquid entirely fills the interior of the smaller ellipsoid, $\mathcal{C}$, described by the second equation in (4.21). Since the liquid is homogeneous, from the results of the previous section we know that $\{O, \boldsymbol{e}_i\}$ is a principal frame of inertia also for $\mathbb{J}_O^*$, with components given in (4.19). Thus, recalling that $\alpha_i = \eta\, a_i$, we find

$$\mathsf{A}^* = \frac{m}{5}\,\eta^2\,\frac{(a_2^2 - a_3^2)^2}{a_2^2 + a_3^2}\,, \quad \mathsf{B}^* = \frac{m}{5}\,\eta^2\,\frac{(a_3^2 - a_1^2)^2}{a_3^2 + a_1^2}\,, \quad \mathsf{C}^* = \frac{m}{5}\,\eta^2\,\frac{(a_1^2 - a_2^2)^2}{a_1^2 + a_2^2}\,,$$

where

$$m = \frac{4}{3}\,\pi\,\rho\,\alpha_1\alpha_2\alpha_3 = \frac{4}{3}\,\pi\,\eta^3\,\rho\,a_1a_2a_3$$

is the mass of the liquid. Accordingly, the principal moments of inertia of the transformed body are then given by

$$\begin{aligned}
\mathsf{A}^1 &:= \mathsf{A} + \mathsf{A}^* = \frac{1}{5}\left[M_{\mathcal{B}}\,\gamma\,(a_2^2 + a_3^2) + m\,\eta^2\,\frac{(a_2^2 - a_3^2)^2}{a_2^2 + a_3^2}\right] \\
\mathsf{B}^1 &:= \mathsf{B} + \mathsf{B}^* = \frac{1}{5}\left[M_{\mathcal{B}}\,\gamma\,(a_1^2 + a_3^2) + m\,\eta^2\,\frac{(a_3^2 - a_1^2)^2}{a_3^2 + a_1^2}\right] \\
\mathsf{C}^1 &:= \mathsf{C} + \mathsf{C}^* = \frac{1}{5}\left[M_{\mathcal{B}}\,\gamma\,(a_1^2 + a_2^2) + m\,\eta^2\,\frac{(a_1^2 - a_2^2)^2}{a_1^2 + a_2^2}\right]
\end{aligned} \tag{4.26}$$

We want to show that we can pick a_1, a_2 and a_3, densities $\rho, \widehat{\rho}$, and thickness η in such a way that (4.24) is still satisfied (so that (4.25) still holds) but

$$\mathsf{A}^1 < \mathsf{C}^1 < \mathsf{B}^1\,. \tag{4.27}$$

This will imply, again by Theorem 3.1.1, that the rotation around $(O, \boldsymbol{e}_3)$ is now *unstable*, while that around $(O, \boldsymbol{e}_2)$ becomes *stable*.

To this end, we choose (for instance)

$$a_2 = \frac{1}{\sqrt{2}}\,a_1\,, \quad a_3 = \varepsilon\,a_1\,,$$

with $\varepsilon \in (0, 1/\sqrt{2})$. Then, after a straightforward calculation, we show that the condition $\mathsf{C}^1 < \mathsf{B}^1$ is equivalent to

$$M_{\mathcal{B}}\,\gamma\,(1 - 2\varepsilon^2) < \frac{m}{3}\,\eta^2\,\frac{6(\varepsilon^2 - 1)^2 - (1 + \varepsilon^2)}{1 + \varepsilon^2}\,,$$

which is certainly satisfied by taking ε sufficiently small, provided

$$m > \frac{3}{5}\frac{\gamma}{\eta^2}\,M_{\mathcal{B}} \iff \rho > \frac{3}{5}\,\frac{1 - \eta^5}{\eta^5}\widehat{\rho}\,. \tag{4.28}$$

In a similar manner, we show that the requirement $\mathsf{A}^1 < \mathsf{C}^1$ is equivalent to

$$M_{\mathcal{B}}\,\gamma\,(\varepsilon^2-1) < \frac{m}{6}\,\eta^2\,\frac{1+\varepsilon^2-3\,(1-2\varepsilon^2)^2}{1+\varepsilon^2}\,,$$

that is certainly satisfied by taking ε small enough, provided

$$m < 3\,\frac{\gamma}{\eta^2}\,M_{\mathcal{B}} \iff \rho < 3\,\frac{1-\eta^5}{\eta^5}\,\widehat{\rho}\,. \tag{4.29}$$

We may thus conclude that condition (4.27) is met in the case at hand, whenever the density of the body and of the liquid satisfy simultaneously (4.28) and (4.29), for a suitable thickness of the shell. For example, if the ellipsoidal shell is glass (density = 2.6 gr/cm^3) and the liquid is water (density = 1.0 gr/cm^3) both requisites are certainly met, provided we choose $\eta \simeq .91$, that is, the shell is sufficiently thin.

Remark 4.4.2. We wish to emphasize that *the inviscid theory, in the class of irrotational flow, is not able to provide a mathematical interpretation of* KELVIN*'s experiment* described in the introductory chapter. Actually, take as body $\mathscr{B}$ a thin spheroidal shell of constant density such that two axes of the spheroid are equal, say, $a_1 = a_2$. In this situation, from (4.26) we obtain $\mathsf{A}^1 = \mathsf{B}^1$ and $\mathsf{C}^1 = \mathsf{C}$. Thus, no matter whether $\mathsf{C}^1 < \mathsf{A}^1$ (prolate spheroid) or $\mathsf{C}^1 > \mathsf{A}^1$ (oblate spheroid), in view of Theorem 4.4.1 the permanent rotation around a_3 is always stable, which is at odds with KELVIN's finding that shows stability only in the latter case.

4.5 Further Stability Results

Results described in Theorem 4.4.1 are obtained under the assumption that the flow of the liquid is irrotational. While this might be satisfactory in the *instability* analysis of a specific steady-state motion, it is no longer so if we are instead interested in its *stability* properties. This because, even though a motion may turn out to be stable in the class of irrotational perturbations, $\mathfrak{I}$, it could still be unstable in a more general class of perturbations that do not satisfy such a condition. This observation then leads us to study the system (2.20)–(2.23), with $\mu = 0$, in its full generality and over the *entire* time interval $(0,\infty)$. However, this would represent a formidable task since, as is well known, already in the case where the motion of the body is prescribed, it is not known whether the initial-boundary value problem for the corresponding (Euler) equations admits a global solution, even of "weak" type. Nevertheless, if we *suppose* that (2.20)–(2.23) admits solutions satisfying, in a suitable sense, balance of energy and (axial) angular momentum for a "sufficiently rich" range of initial data, we can draw a number of interesting conclusions that, as we shall analyze in full details later on, will be very useful also in understanding the effect of viscosity in analogous stability questions; see Remark 4.5.4.

With this in mind, we begin to consider the case when the body is heavy and constrained to move around a fixed point O. As in Section 3.2.1.2, we assume that the center of mass G of $\mathscr{S}$ lies on one of the axes, $(O, \boldsymbol{e}_3)$ say, of the principal frame of inertia $\{O, \boldsymbol{e}_i\}$. Without loss of generality, we orient $\{O, \boldsymbol{e}_i\}$ in such a way that $\boldsymbol{x}_G = z_0\, \boldsymbol{e}_3$, $z_0 > 0$. In this a case, (2.20)–(2.23) along with $(3.19)_2$ and (3.20) furnish

$$\left.\begin{aligned} &\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \dot{\boldsymbol{\omega}}\times\boldsymbol{x} + 2\boldsymbol{\omega}\times\boldsymbol{v}\Big) = -\nabla \mathsf{p} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\}\quad (x,t)\in\mathcal{C}\times(0,\infty)$$
$$\boldsymbol{v}(x,t)\cdot\boldsymbol{n} = \boldsymbol{0}\,,\quad (x,t)\in\partial\mathcal{C}\times(0,\infty) \tag{4.30}$$

and

$$\begin{gathered} \dot{\boldsymbol{K}}_O + \boldsymbol{\omega}\times\boldsymbol{K}_O = \beta^2\boldsymbol{e}_3\times\boldsymbol{\gamma}\,,\quad \boldsymbol{K}_O := \mathbb{I}_O\cdot\boldsymbol{\omega} + \int_{\mathcal{C}}\rho\,\boldsymbol{x}\times\boldsymbol{v}\,, \\ \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \boldsymbol{0} \end{gathered} \tag{4.31}$$

with $\mathbb{I}_O$ and β^2 given in (2.23) and (3.25). Following [6], we introduce the new variable

$$\boldsymbol{\omega}_* := \mathbb{I}_O^{-1}\cdot\boldsymbol{K}_O \equiv \boldsymbol{\omega} + \mathbb{I}_O^{-1}\cdot\left(\int_{\mathcal{C}}\rho\,\boldsymbol{x}\times\boldsymbol{v}\right)\,, \tag{4.32}$$

so that the preceding equations become

$$\left.\begin{aligned} &\rho\Big(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_* + \dot{\boldsymbol{a}})\times\boldsymbol{x} + 2(\boldsymbol{\omega}_* + \boldsymbol{a})\times\boldsymbol{v}\Big) = -\nabla \mathsf{p} \\ &\operatorname{div}\boldsymbol{v} = 0 \end{aligned}\right\}\ \text{in}\ \ \mathcal{C}\times(0,\infty)$$
$$\boldsymbol{v}(x,t)\cdot\boldsymbol{n} = \boldsymbol{0}\,,\quad \text{at } \partial\mathcal{C}\times(0,\infty) \tag{4.33}$$

and

$$\left.\begin{aligned} \mathbb{I}\cdot\dot{\boldsymbol{\omega}}_* + (\boldsymbol{\omega}_* + \boldsymbol{a})\times(\mathbb{I}\cdot\boldsymbol{\omega}_*) = \beta^2\boldsymbol{e}_3\times\boldsymbol{\gamma} \\ \dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}_* + \boldsymbol{a})\times\boldsymbol{\gamma} = \boldsymbol{0} \end{aligned}\right\}\ \text{in } (0,\infty)\,, \tag{4.34}$$

with $\mathbb{I} \equiv \mathbb{I}_O$, and

$$\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{v}) := -\mathbb{I}^{-1}\cdot\left(\int_{\mathcal{C}}\rho\,\boldsymbol{x}\times\boldsymbol{v}\right)\,. \tag{4.35}$$

Now, the remarkable feature is that equations (4.33)–(4.35), in addition to the geometric constraint (3.22), possess two first integrals (conservation laws) that are (formally) analogous to (3.20) and (3.21). To see this, let us dot-multiply both sides of $(4.33)_1$ by $\boldsymbol{v}$ and integrate by parts over $\mathcal{C}$. We thus have, with the help of $(4.33)_{2,3}$,

$$\tfrac{1}{2}\,\rho\,\frac{d}{dt}\|\boldsymbol{v}(t)\|_2^2 = -(\dot{\boldsymbol{\omega}}_* + \dot{\boldsymbol{a}})\cdot\int_{\mathcal{C}}\rho\boldsymbol{x}\times\boldsymbol{v}\,. \tag{4.36}$$

Recalling that $\mathbb{I}$ is symmetric, from (4.34) and (4.35) we infer

$$-\dot{\boldsymbol{\omega}}_* \cdot \int_{\mathcal{C}} \rho \boldsymbol{x} \times \boldsymbol{v} = \boldsymbol{a} \cdot \mathbb{I} \cdot \dot{\boldsymbol{\omega}}_* = -\boldsymbol{\omega}_* \times (\mathbb{I} \cdot \boldsymbol{\omega}_*) \cdot \boldsymbol{a} + \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma} \cdot \boldsymbol{a} \,. \tag{4.37}$$

Furthermore, by dot-multiplying both sides of $(4.34)_1$ by $\boldsymbol{\omega}_*$ and using $(4.34)_2$, we get

$$\tfrac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}_* \cdot \mathbb{I} \cdot \boldsymbol{\omega}_*) - \beta^2\, \dot{\gamma}_3 = \boldsymbol{\omega}_* \times (\mathbb{I} \cdot \boldsymbol{\omega}_*) \cdot \boldsymbol{a} - \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma} \cdot \boldsymbol{a} \,,$$

whereas, by (4.35),

$$-\dot{\boldsymbol{a}} \cdot \int_{\mathcal{C}} \rho \boldsymbol{x} \times \boldsymbol{v} = \tfrac{1}{2}\frac{d}{dt}(\boldsymbol{a} \cdot \mathbb{I} \cdot \boldsymbol{a}) \,.$$

Combining the last two displayed equations with (4.36) and (4.37), we (formally) conclude

$$\mathcal{E}(t) - \mathcal{U}(t) = \mathcal{E}(0) - \mathcal{U}(0) \,, \quad \text{all } t \in [0,\infty) \,, \tag{4.38}$$

where

$$\begin{aligned} &\mathcal{E} := \tfrac{1}{2}\left(\rho\, \|\boldsymbol{v}\|_2^2 - \boldsymbol{a} \cdot \mathbb{I} \cdot \boldsymbol{a} + \boldsymbol{\omega}_* \cdot \mathbb{I} \cdot \boldsymbol{\omega}_*\right) \,, \\ &\mathcal{U} := \beta^2\, \gamma_3 \,. \end{aligned} \tag{4.39}$$

In addition to the first integral (4.38) we shall now deduce another one concerning the component of the angular momentum along the vertical direction. In fact, if we dot-multiply first both sides of $(4.34)_1$ by $\boldsymbol{\gamma}$, then both sides of $(4.34)_2$ by $\mathbb{I} \cdot \boldsymbol{\omega}_*$, and afterword sum side-by-side the resulting equations, we get

$$\boldsymbol{\gamma}(t) \cdot \mathbb{I} \cdot \boldsymbol{\omega}_*(t) = \boldsymbol{\gamma}(0) \cdot \mathbb{I} \cdot \boldsymbol{\omega}_*(0) \,. \tag{4.40}$$

Finally, from $(4.34)_2$ we also deduce

$$\gamma_1^2(t) + \gamma_2^2(t) + \gamma_3^2(t) = 1 \,. \tag{4.41}$$

Let us indicate by $\mathfrak{C}$ the class of solutions $(\boldsymbol{\omega}_*, \boldsymbol{\gamma}, \boldsymbol{v})$ to (4.33)–(4.34) that satisfy (4.38), (4.40) and (4.41). Clearly, $\mathfrak{C}$ is not empty, since it contains the class $\mathfrak{I}$ of irrotational flow

$$\mathfrak{I} := \big\{(\boldsymbol{\omega}_*, \boldsymbol{u} - (\boldsymbol{\omega}_* + \boldsymbol{a}) \times \boldsymbol{x}, \boldsymbol{\gamma}) : \ \boldsymbol{u} = \nabla\varphi\big\}$$

Our objective is to investigate the stability of some relevant steady-state motions in the class $\mathfrak{C}$. To this end, set

$$\mathsf{S}_0 := \big\{(\boldsymbol{\omega}_*, \boldsymbol{\gamma}, \boldsymbol{v} \equiv \boldsymbol{0}) \,, \quad (\boldsymbol{\omega}_*, \boldsymbol{\gamma}) \in \mathrm{S}_0 := \mathsf{PR} \cup \mathsf{SP}_0 \cup \mathsf{SP}_1 \cup \mathsf{SP}_2\big\},$$

with PR and SP_i, $i = 0,1,2$, defined in (3.28)–(3.30) by replacing A, B, and C with A, B, and C, respectively. Taking into account the results of Section 3.2.1.1, one immediately checks that every $\mathsf{s}_0 \in \mathsf{S}_0$ is a steady-state solutions to (4.33)–(4.34), representing a uniform rotation around the vertical axis through O, or else a steady precession of the coupled system $\mathscr{S}$ as a whole, with the liquid being at (relative) rest; see Section 3.2.1.1.

Remark 4.5.1. It is somehow relevant to observe that the class S_0 is a *strict* subset of the class of all possible (and "sufficiently smooth") time-independent solutions to (4.32)–(4.35). In fact, denote by S_0^1 the class S_0 with the replacement $(\mathsf{A},\mathsf{B},\mathsf{C}) \to (\mathsf{A}^1,\mathsf{B}^1,\mathsf{C}^1)$. Then, from the results of Section 4.2, it follows that any element in the set

$$\mathsf{S}_0^1 := \big\{\big(\boldsymbol{v}_0 := \nabla(\omega_{0i}\,\psi_i) - \boldsymbol{\omega}\times\boldsymbol{x}, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0\big) :\ (\boldsymbol{\omega}_0,\boldsymbol{\gamma}_0) \in \mathrm{S}_0^1\,,\ \psi_i \text{ solution to } (4.8)_2\big\}$$

is a steady-state solution to (4.30)–(4.31), and therefore to (4.33)–(4.35) after taking into account the definition (4.32). Notice that S_0^1 coincides with S_0 only in those cases when $\psi_i \equiv 0$, $i = 1,2,3$. However, as we will see in a later chapter, in the case of a *viscous* liquid, S_0 exhausts the *entire class* of time-independent motions.

A crucial point in the stability analysis we are about to perform is that the functional $(4.39)_1$ is positive definite in the variables $(\boldsymbol{v},\boldsymbol{\omega}_*)$. In fact, this property is a consequence of the following result, whose first part is due to Kopachevsky and Krein [8, §§7.2.2–7.2.4].

Lemma 4.5.2. *Let $\boldsymbol{w} \in L^2(\mathcal{C})$. Then, there is $c_0 = c_0(\rho,\mathcal{C}) \in (0,1)$ such that*

$$\|\boldsymbol{w}\|_2^2 \geq \|\boldsymbol{w}\|_2^2 - \left(\rho\,\mathbb{I}^{-1}\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{w}\right)\cdot\left(\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{w}\right) \geq c_0\,\|\boldsymbol{w}\|_2^2\,.$$

We are now in a position to investigate the stability property of steady-state motions in the class $\mathfrak{C}$. In this regard, we premise the following.

Definition 4.5.3. An element $\mathsf{s}_0 \in \mathsf{S}_0$ is said to be *stable* (in the class $\mathfrak{C}$) if, denoting by $\mathsf{s}_0 + (\boldsymbol{\zeta}(t),\boldsymbol{z}(t),\boldsymbol{v}(t))$ an element in the class $\mathfrak{C}$, it happens that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{v}(0)\|_2 + |\boldsymbol{\zeta}(0)| + |\boldsymbol{z}(0)| < \delta \quad\Longrightarrow\quad \sup_{t\geq 0}\left(\|\boldsymbol{v}(t)\|_2 + |\boldsymbol{\zeta}(t)| + |\boldsymbol{z}(t)|\right) < \varepsilon\,.$$

Remark 4.5.4. Notice that the stability property of the liquid is measured in terms of its kinetic energy.

The following result, a variant of Proposition 3.0.1, furnishes sufficient conditions for the stability of a steady-state solution s_0 in the class $\mathfrak{C}$.

Proposition 4.5.5. *Let $\mathsf{s}_0 \in \mathsf{S}_0$ and set $\boldsymbol{y} := (\boldsymbol{\zeta},\boldsymbol{z})$. Moreover, let $F : L^2(\mathcal{C}) \to [0,\infty)$ be such that*

$$c_1\|\boldsymbol{v}\|_2 \leq F(\boldsymbol{v}) \leq c_2\|\boldsymbol{v}\|_2\,, \tag{4.42}$$

and let $\mathsf{U} :\ \boldsymbol{y} \in \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ be continuous and, in addition, positive definite in a neighborhood $\mathcal{I}(\mathbf{0})$ of the origin of $\mathbb{R}^3\times\mathbb{R}^3$, namely,

(i) $\mathsf{U}(\mathbf{0}) = 0$,

(ii) $\mathsf{U}(\boldsymbol{y}) > 0$ *for all* $\boldsymbol{y} \in \mathcal{I}(\mathbf{0}) \setminus \{\mathbf{0}\}$.

Then, if $\mathsf{V}(t) := F(\boldsymbol{v}(t)) + \mathsf{U}(\boldsymbol{y}(t))$ *satisfies* $\mathsf{V}(t) \leq \mathsf{V}(0)$ *in the class* $\mathfrak{C}$ *for all* $t \geq 0$, s_0 *is stable.*

Proof. Denote by $[\boldsymbol{y}] := \sqrt{\boldsymbol{\zeta}^2 + \boldsymbol{z}^2}$ the Euclidean norm in $\mathbb{R}^3 \times \mathbb{R}^3$. Next, Let $\varepsilon_0 > 0$ be such that $B_{\varepsilon_0}(\mathbf{0}) \subset \mathcal{I}(\mathbf{0})$. Fix $\varepsilon \in (0, \varepsilon_0)$ and set

$$\xi := \min_{[\boldsymbol{y}]=\varepsilon/2} \mathsf{U}(\boldsymbol{y}) > 0.$$

The minimum exists since U is continuous on $B_{\varepsilon_0}(\mathbf{0})$ and the sphere of radius ε is compact in $\mathbb{R}^3$. Moreover, ξ is strictly positive because of condition (ii). Again by the continuity of U, we find $\delta_0 > 0$ such that $[\boldsymbol{y}] < \delta_0$ implies $\mathsf{U}(\boldsymbol{y}) < \frac{1}{2}\min\{\xi,\ \varepsilon, \frac{1}{2}c_1\varepsilon\}$, with c_1 defined in (4.46). Choose $\delta < \min\{\delta_0, \frac{1}{2c_2}\xi, \frac{1}{4}\frac{c_1}{c_2}\varepsilon\}$ with c_2 as in (4.46). We want to show that

$$[\boldsymbol{y}(t)] < \varepsilon/2\,, \quad \text{for all } t \geq 0. \tag{4.43}$$

Suppose, on the contrary, that $\bar{t}$ is the first instant of time when $[\boldsymbol{y}(\bar{t})] = \varepsilon/2$. Thus, since F is positive definite and $\mathsf{V}(t) \leq \mathsf{V}(0)$ for all $t \geq 0$, we deduce with the help of (4.46)

$$\xi \leq \mathsf{V}(\bar{t}) = F(\boldsymbol{v}(\bar{t})) + \mathsf{U}(\boldsymbol{y}(\bar{t})) \leq c_2\|\boldsymbol{v}(0)\| + \mathsf{U}(\boldsymbol{y}(0)) < \frac{\xi}{2} + \frac{\xi}{2} = \xi\,,$$

which shows a contradiction. Thus, (4.47) holds and, in addition, for all $t \geq 0$,

$$c_1\|\boldsymbol{v}(t)\| \leq F(\boldsymbol{v}(t)) \leq \mathsf{V}(0) \leq c_2\|\boldsymbol{v}(0)\| + \mathsf{U}(\boldsymbol{y}(0)) < \tfrac{1}{4}c_1\varepsilon + \tfrac{1}{4}c_1\varepsilon = \tfrac{1}{2}c_1\varepsilon$$

which completes the proof.

□

The remarkable feature of this proposition consists in the fact that, as we shall show shortly, it allows us to prove *verbatim* most of the *stability* results established in Section 3.1.1 also when the cavity is filled with an inviscid liquid, and in a class of perturbations, in principle, more general than that of irrotational flow, which only requires the validity of the basic physical principles of conservations of energy and axial angular momentum. To this end, *it will be enough to formally replace the energy function* $T - U$ *in that section with the energy functional* $\mathcal{E} - \mathcal{U}$ *defined in* (4.39).

As a way of example, let us take $\mathsf{s}_0 = (r_0\,\boldsymbol{e}_3, -\boldsymbol{e}_3, \mathbf{0})$ (Upright Spinning Top). From (4.38)–(4.41) we deduce that the following three functions are constant along solutions in the class $\mathfrak{C}$

$$\begin{aligned}
\mathsf{V}_1 &:= 2E + A\,\zeta_1^2 + B\,\zeta_2^2 + C\,(\zeta_3^2 + 2r_0\zeta_3) - 2\beta^2\,z_3\,,\\
\mathsf{V}_2 &:= A\,\zeta_1 z_1 + B\,\zeta_2 z_2 + C\,(\zeta_3 z_3 + r_0 z_3 - \zeta_3)\\
\mathsf{V}_3 &:= z_1^2 + z_2^2 + z_3^2 - 2z_3,
\end{aligned} \tag{4.44}$$

where A, B and C are the principal moments of inertia of $\mathbb{I}$ in $\{O, \boldsymbol{e}_i\}$, and

$$E := \tfrac{1}{2}\rho\,\|\boldsymbol{v}\|_2^2 - \tfrac{1}{2}\boldsymbol{a}\cdot\mathbb{I}\cdot\boldsymbol{a}. \tag{4.45}$$

Suppose, now, as in Theorem 4.5.6, that $A \le B < C$ (Generic Top) and define, in entirely analogy with (3.42),

$$\mathsf{V} = \mathsf{V}_1 + 2\,r_0\,\mathsf{V}_2 + (C\,r_0^2 - \beta^2)\mathsf{V}_3 + \eta\,\mathsf{V}_3^2\,, \quad \eta > 0.$$

Obviously, $\mathsf{V}(t) = \mathsf{V}(0)$ in the class $\mathfrak{C}$. Furthermore from (4.48) and (3.39)–(3.41) we obtain

$$\mathsf{V} = 2E + V\,,$$

where V is defined in (3.39). Now, by Lemma 4.5.2, $F \equiv 2E$ satisfies the assumption of Proposition 4.5.5. On the other hand, from the proof of Theorem 4.5.6 we know that if

$$r_0^2 > \frac{\beta^2}{C - M}\,, \quad M := \max\{A, B\}\,, \tag{4.46}$$

the function $\mathsf{U} \equiv V$ is (continuous) and positive definite in a neighborhood of $\boldsymbol{\zeta} = \boldsymbol{z} = \boldsymbol{0}$. As a result, we conclude with the following.

Theorem 4.5.6. (Upright Spinning Top) *Let* $\mathsf{s}_0 = (r_0\,\boldsymbol{e}_3, -\boldsymbol{e}_3, \boldsymbol{0})$. *Then, if* (4.50) *holds,* s_0 *is stable in the class* $\mathfrak{C}$.

By a similar argument, one can show stability results in the class $\mathfrak{C}$, entirely analogous to those of Theorem 3.2.5 (Hanging Spinning Top) and Theorem 3.2.6–Theorem 3.2.7 (Steady Precession), with the only replacement $(\mathsf{A}, \mathsf{B}, \mathsf{C}) \to (A, B, C)$.

Remark 4.5.7. We would like to notice that, however, the method just described does not allow us to extend to the case at hand the result showed in Theorem 3.2.2, when the cavity is empty, regarding the *stability* of the spinning symmetric top (Lagrange Top). The reason is due to the fact that, when the cavity is filled with the liquid, the component ω_3 (or ω_{*3}) is, in general, no longer a first integral, as it is at once recognized by taking the third component of equation $(4.34)_1$ with $A = B$.

Proposition 4.5.5 can also be employed to investigate the stability properties of permanent rotations in the case of *inertial motions*. In this situation the conservation of energy (4.38) becomes

$$E(t) + \tfrac{1}{2}\,\boldsymbol{\omega}_*(t)\cdot\mathbb{I}\cdot\boldsymbol{\omega}_*(t) = E(0) + \tfrac{1}{2}\,\boldsymbol{\omega}_*(0)\cdot\mathbb{I}\cdot\boldsymbol{\omega}_*(0)\,, \quad \text{all } t \in [0, \infty)\,, \tag{4.47}$$

where E is defined in (4.50) and $\mathbb{I} \equiv \mathbb{I}_G$. In addition, by dot-multiplying both sides of (4.34) by $\mathbb{I}\cdot\boldsymbol{\omega}_*$ we obtained another first integral:

$$|\mathbb{I}\cdot\boldsymbol{\omega}_*(t)|^2 = |\mathbb{I}\cdot\boldsymbol{\omega}_*(0)|^2\,.$$

By choosing the body-fixed frame coinciding with the central frame of inertia of the *coupled* system liquid-body $\mathscr{S}$, we infer that the previous relation can be written as

$$A^2\, p_*^2(t) + B^2\, q_*^2(t) + C^2\, r_*^2(t) = A^2\, p_*^2(0) + B^2\, q_*^2(0) + C^2\, r_*^2(0)\,, \tag{4.48}$$

where A, B and C are the central moments of inertia of $\mathscr{S}$ and $\boldsymbol{\omega}_* = (p_*, q_*, r_*)$.

As before, let us indicate by $\mathfrak{C}$ the class of solutions $(\boldsymbol{\omega}, \boldsymbol{v})$ to (4.33)–(4.34) that satisfy (4.51) and (4.52), and let $\mathsf{s}_0 := (\omega_0\, \boldsymbol{e}, \boldsymbol{v} \equiv \mathbf{0})$, $\omega_0 \neq 0$, $\boldsymbol{e} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, be a given permanent rotation.

We shall say that s_0 is *stable* (in the class $\mathfrak{C}$) if, denoting by $(\omega_0\, \boldsymbol{e} + \boldsymbol{\zeta}(t), \boldsymbol{v}(t))$ an element in the class $\mathfrak{C}$, it happens that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{v}(0)\|_2 + |\boldsymbol{\zeta}(0)| < \delta \quad \Longrightarrow \quad \sup_{t \geq 0} \left(\|\boldsymbol{v}(t)\|_2 + |\boldsymbol{\zeta}(t)|\right) < \varepsilon\,. \tag{4.49}$$

We shall now employ Proposition 4.5.5 (with $\boldsymbol{z} \equiv \mathbf{0}$) to study the stability of the permanent rotation around the central axis of *larger* moment of inertia. So, to fix the ideas, assume $A \leq B < C$ and take $\mathsf{s}_0 = (r_0\, \boldsymbol{e}_3, \boldsymbol{v} \equiv \mathbf{0})$. From (4.51)–(4.52) we get

$$\begin{aligned} \mathsf{E} &:= 2E + A\, \zeta_1^2(t) + B\, \zeta_2^2(t) + C\, \zeta_3^2(t) + 2C\, r_0\, \zeta_3(t) = \text{const.} \\ \mathsf{K} &:= A^2\, \zeta_1^2(t) + B^2\, \zeta_2^2(t) + C^2\, \zeta_3^2(t) + 2\, C^2\, r_0\, \zeta_3(t) = \text{const.}. \end{aligned} \tag{4.50}$$

Mimicking the choice (3.13), we pick

$$\mathsf{V} := C\, \mathsf{E} - \mathsf{K} + \frac{1}{4\, r_0^2\, C^2}\, \mathsf{K}^2\,. \tag{4.51}$$

If we now replace in (4.55) the expressions for E and K, we infer

$$\begin{aligned} \mathsf{V} =&\; 2CE(\boldsymbol{v}) + \mathsf{U}(\boldsymbol{\zeta})\,, \\ \mathsf{U}(\boldsymbol{\zeta}) :=&\; A(C-A)\, \zeta_1^2 + B(C-B)\, \zeta_2^2 + C^2 \zeta_3^2 + f(\boldsymbol{\zeta})\,, \end{aligned} \tag{4.52}$$

where f is such that, for a given arbitrary $\eta > 0$, $|f| < \eta\, |\boldsymbol{\zeta}|^2$ whenever $|\boldsymbol{\zeta}| < c\, \eta$. Now, from (4.54) and (4.55) it follows that the functional V is constant in the class $\mathscr{C}$. Moreover, the function U is (continuous) and positive definite in a neighborhood of $\boldsymbol{\zeta} = \mathbf{0}$ provided

$$C > B \geq A\,.$$

By Proposition 4.5.5 (with $\boldsymbol{z} \equiv \mathbf{0}$), (4.50), and Lemma 4.5.2 we then conclude with the following.

Theorem 4.5.8. (Stability of Permanent Rotations in Inertial Motions) *Permanent rotations around the the axis of maximum moment of inertia are stable in the class* $\mathfrak{C}$.

Remark 4.5.9. As already observed, the stability result in Theorem 4.5.8 is, in principle, more general than the analogous one stated in Theorem 4.4.1 for the case of inertial motions, because it is proved in a class $\mathfrak{C}$ where the perturbed liquid flow need not be irrotational, but only satisfy the physical principles of conservation of energy and angular momentum. In this regard and with a view to KELVIN's experiment (see Remark 4.4.2), it is important to emphasize that in the class $\mathfrak{C}$, where vorticity is relevant, one is *not* able to show that permanent rotations around the axis of *minimum* moment of inertia are stable. The reason for this relies on the circumstance that if we mimic the choice (3.11) by replacing $2T$ with E, the corresponding Lyapunov function would no longer be positive definite, due to the presence of the term $-2\,A\,E$ which is *negative* and cannot be balanced by any other quantity. This fact suggests that vorticity may play an important role in the explanation of the experiment. Actually, we shall show in a later chapter that if the liquid is taken to be *viscous*, the mathematical analysis will entirely support KELVIN's experimental finding.